PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 6 (20), 1966. p. 121 — 123

THE SUBSETS P^+ AND P^- OF THE SPLIT INTERVAL

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(Presented, December 10, 1965)

Definition 1. Let $u=\{t|0<t<1\}$, $v=\{0,1\}$ and let these sets be ordered in the natural way. Put

$$P=\{[0,1]\}\oplus(u\circ v)\oplus\{[1,0]\},$$

where $\oplus$ denotes Birkhoff's ordinal sum and $\circ$ Birkhoff's ordinal product ([1]). The set P is a chain with respect to this ordering. *The split interval* is P together with the interval topology with respect to this ordering.

Analogously as in [2] denote

$$P^+=\{[\xi, 1]\,|\,0\leqq\xi<1\},$$

$$P^-=\{[\xi, 0]\,|\,0<\xi\leqq 1\},$$

$$M(A)=\{x\,|\,[x, 0]\in A, [x, 1]\in A\},$$

$$M^+(A)=\{x\,|\,[x, 0]\notin A, [x, 1]\in A\},$$

$$M^-(A)=\{x\,|\,[x, 0]\in A, [x, 1]\notin A\},$$

for $A\subseteq P$.

Furthermore, denote by $\mathfrak{N}$ the system of all infinite sequences of natural numbers. By *an open interval* in an ordered set X with the smallest element o and the greatest element i we understand a non-empty set $\{t\,|\,t\in X, a<t<b\}$ or $\{t\,|\,t\in X, o\leqq t<b\}$ or $\{t\,|\,t\in X, a<t\leqq i\}$ or $\{t\,|\,t\in X, o\leqq t\leqq i\}=X$, where $a\in X$, $b\in X$. The elements *a,b or o,b or a,i or o,i* will be called *the end-points* of corresponding open intervals.

Definition 2. Let $\mathfrak{M}\neq\varnothing$ be any system of sets. Assign to every finite sequence of natural numbers $(n_1, n_2,\ldots, n_k)$ a set $M_{n1, n2,\ldots, nk}\in\mathfrak{M}$. Put for any infinite sequence $\nu=(n_1, n_2, \ldots)$ of natural numbers

$$M_\nu=M_{n_1}\cap M_{n_1, n_2}\cap M_{n_1, n_2, n_3}\cap\ldots .$$

Then, the set $Z=U\,M_\nu$ $(\nu\in\mathfrak{A})$ is called a *Suslin set upon the system* $\mathfrak{M}$.

Prof. K. Koutský showed in his lecture [2], at the conference on ordered sets in November 1963 in Brno that sets P^+ and P^- are not Borel sets.

Đ. Kurepa put then the question, whether these sets are Suslin sets. In this paper, there is shown that the sets P^+ and P^- are neither Suslin sets upon the system of all open sets nor Suslin sets upon the system of all closed sets of the split interval P.

Lemma. *A set $\varnothing \neq A \subseteq P$ is open if $A = \bigcup_{1 \leq n^2 \alpha} I_n$, where I_n are mutually disjoint open intervals in P and $1 < \alpha \leq \omega$.*

Proof. The sufficiency of this condition is clear. Assume therefore that $\varnothing \neq A \subseteq P$ is open. Clearly $\varnothing \neq M(A)$ is an open set in $J = \{t \mid 0 \leq t \leq 1\}$, hence $M(A) = \bigcup_{1 \leq n < \alpha} J_n$, where $1 < \alpha \leq \omega$ and J_n are mutually disjoint open intervals in J, hence $J_n = \{t \mid a_n < (\leq)\ t < (\leq)\, b_n\}$, where $0 \leq a_n < b_n \leq 1$. Denote $I_n = (\{[\xi, 0] \mid \xi \in J_n\} \cup \{[\xi, 1] \mid \xi \in J_n\} \cup \{[a_n, 1], [b_n, 0]\}) \cap A$. I_n are disjoint open intervals in P for $1 \leq n < \alpha$ and $I_n \subseteq A$.

Let $[a, 1] \in A$. As A is open in P, there exists a real number b such that $a < b \leq 1$ and $\{t \mid a < t < b\} \subseteq J_{n0}$. Then there exists a natural number n_0 such that $\{t \mid a < t < b\} \subset I_{n0}$. This implies $[a, 1] \in I_{n0}$. Dually it holds that for $[a, 0] \in A$ there exists a natural number n_0' such that $[a, 0] \in I'_{n_0}$. Thus $\bigcup_{\alpha \leq n < 1} I_n = A$.

Theorem 1. *Let $A \subseteq P$ be a Suslin set upon the system of all open sets in P. Then card $M^+(A)$ + card $M^-(A) \leq \aleph_0$.*

Proof. Let $A \subseteq P$ be a Suslin set upon the system of all open sets in P. Then to every finite sequence of natural numbers $(n_1, \ldots, n_k)$ there exists an open set $M_{n_1, \ldots, n_k}$ such that $A = \bigcup M_\nu$ $(\nu \in \mathfrak{N})$ and for $\nu = (n_1, n_2, \ldots)$ we have $M_\nu = M_{n_1} \cap M_{n_1, n_2} \cap \ldots$. According to Lemma for every finite sequence of natural numbers $(n_1, \ldots, n_k)$ we have $M_{n_1, \ldots n_k} = \varnothing$ or $M_{n_1, \ldots n_k} = \bigcup_{1 \leq n < 2\varkappa n_1, \ldots, n_k} I_n^{n_1, \ldots, n_k}$, where $1 < \alpha_{n_1, \ldots, n_k} \leq \omega$ and $I_n^{n_1, \ldots n_k}$ are mutually disjoint open intervals for $1 \leq n < \alpha_{n_1, \ldots, n_k}$.

If $M_{n_1, \ldots, n_k} \neq \varnothing$, denote $K_{n_1, \ldots, n_k}$ the set of all end-points of intervals $I_n^{n_1, \ldots, n_k}$, $1 \leq n < \alpha_{n_1, \ldots, n_k}$. If $M_{n_1, \ldots, n_k} = \varnothing$, put $K_{n_1, \ldots, n_k} = \varnothing$. We have card $K_{n_1, \ldots, n_k} \leq \aleph_0$. Denote $K = \bigcup K_\varkappa$, where $\varkappa$ runs over all finite sequences of natural numbers. Then card $K \leq \aleph_0$.

Let $a \in M^+(A)$. Then there exists $\nu_0 \in \mathfrak{N}$, $\nu_0 = (n_1^\circ, n_2^\circ, \ldots)$ such that $[a, 1] \in M_{\nu_0}$. As $[a, 0] \notin A$, there exists a natural number k such that $[a, 0] \notin M_{n_1^\circ, \ldots, n_k^\circ}$. If $a = 0$, then $[0,1] \in K_{n_1^\circ, \ldots, n_k^\circ}$. If $a \neq 0$, then $[a, 0] \in K_{n_1^\circ, \ldots, n_k^\circ}$. Hence card $M^+(A) \leq$ card $K \leq \aleph_0$. Dually we have card $M^-(A) \leq \aleph_0$.

Theorem 2. *Let $A \subseteq P$ be a Suslin set upon the system of all closed sets in P. Then card $M^+(A)$ + card $M^-(A) \leq \aleph_0$,*

Proof. Let $A \subseteq P$ be a Suslin set upon the system of all closed sets in P. Then to every finite sequence of natural numbers $(n_1, \ldots, n_k)$ there exists a closed set $M'_{n_1, \ldots, n_k}$ such that $A = \bigcup M'_\nu$ $(\nu \in \mathfrak{N})$ and for $\nu = (n_1, n_2, \ldots)$ we have $M'_\nu = M'_{n_1} \cap M'_{n_1, n_2} \cap \ldots$. Put $M_{n_1, n_2, \ldots, n_k} = P - M'_{n_1, n_2, \ldots n_k}$ for every finite sequence of natural numbers $(n_1, \ldots, n_k)$. The sets $M_{n_1, \ldots, n_k}$ are open and define the set K in the same way like in the proof of Theorem 1.

Let $a \in M^+(A)$. There exits $\nu_0 \in \mathfrak{N}$, $\nu_0 = (n_1^0, n_2^0, \ldots)$ such that $[a, 1] \in M'_{\nu_0}$. As $[a, 0] \notin A$, there exists a natural number k such that $[a, 0] \notin M'_{n_1^0, \ldots, n_k^0}$. If

$a \neq 0$, then $[a, 0] \in M_{n_1^0, \ldots, n_k^0}$, $[a, 1] \notin M_{n_1^0, \ldots, n_k^0}$. From this follows $[a, 1] \in K$ in the case that $a \neq 0$. Hence card M^+ $(A) \leqq$ card $K+1 \leqq \aleph_0$. Dually we have card M^- $(A) \leqq \aleph_0$.

From Theorems 1 and 2 there follows

Theorem 3. *The set P^+ and P^- are neither Suslin sets upon the system of all open sets nor Suslin sets upon the system of all closed sets of the split interval P,*

Note. Professor Đ. Kurepa (in his letter to me) has put the problem whether the sets P^+ and P^- are projective.

REFERENCES

[1] G. Birkhoff *Lattice Theory*, rev. ed., New York, 1948.
[2] K. Koutský *Über das gespaltene Interval.* Publ. Fac. Sci. Univ. Brno, No 457 (1964 474-475.