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NOTE ON RECURRENCE RELATIONS FOR STIRLING NUMBERS

*H. W. Gould**

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A popular notation for Stirling numbers of the first kind is that used by Jordan [7] who introduced the symbol S_n^k so that

$$(1)\qquad (x)_n = x(x-1)\cdots(x-n+1) = n!\binom{x}{n} = \sum_{k=0}^{n} S_n^k\, x^k,$$

where we also suppose that $S_0^0 = 1, S_n^0 = 0\ (n \geqslant 1),\ S_n^k = 0\ (k > n)$.

In this context, Mitrinović and Đoković [8] have found the recurrence formula

$$(2)\qquad (-1)^{n-r}\, S_n^r = \frac{1}{n-r}\sum_{k=0}^{n-r-1}(-1)^k\binom{n-k}{r-1} S_n^{n-k} \qquad (r < n).$$

It may be of interest to point out that this formula occurs in disguised form in the extensive compendium of formulas put together by Hagen [6] in 1891, drawn from earlier sources. Not only this, but we will show that a similar formula exists for the Stirling numbers of the second kind.

The literature relating to the Stirling numbers is afflicted with the disease of multiple notations, which difficulty makes it very troublesome to check relations because of the fact that at least fifty or more notations have been popular. New notations are being introduced all the time. The reader may refer to items in our appended list of references for some of the historical references about such notations. In particular, Hagen [6, p. 60 et seq.] defined the symbol "$[C_k(n)]$" to be the „Combinationen *ohne* Wiederholung" or sum of the $\binom{n}{k}$ possible products, each with k different factors, which may be formed from the first n natural numbers. Cf. Nielsen [9, p. 68]. For this number we shall use the notation $S_1(n, k)$ introduced by the writer in 1960 [1]. It then follows also that

$$(3)\qquad \prod_{k=0}^{n}(1+kx) = \sum_{k=0}^{n} S_1(n, k)\, x^k.$$

The notations of Riordan, Jordan, and the author are related as follows:

$$(4)\qquad s(n, k) = S_n^k = (-1)^{n-k}\, S_1(n-1,\ n-k).$$

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Hagen also defined „Combinationen *mit* Wiederholung" using the special symbol "$\left[\overset{w}{C}_k(n)\right]$" to mean the sum of the $\binom{n+k-1}{k}$ possible products, each with k factors (allowing repetition of factors), which may be formed from the first n natural numbers. For this we shall use the suggestive symbol $S_2(n, k)$, as these are really Stirling numbers of the second kind, also called differences of zero. Here the notations of Riordan, Jordan, and the author are related as follows:

(5) $$S(n, k) = \mathfrak{S}_n^k = S_2(k, n-k) = \frac{1}{k!} \Delta^k 0^n.$$

Corresponding to (3) we now have

(6) $$\prod_{k=0}^{n} (1-kx)^{-1} = \sum_{k=0}^{\infty} S_2(n, k)\, x^k.$$

Hagen notes a very curious pair of relations which relate the Stirling numbers of first and second kind:

(7) $$S_1(n, k) = S_2(-n-1, k),$$

and

(8) $$S_2(n, k) = S_1(-n-1, k).$$

These may be obtained from a more general polynomial of form $S_1(z, k)$, by extending the combinatorial meaning to cases where z is no longer an integer.

Hagen [6, p. 60] gives the following relations

(9) $$S_1(n, k) = \frac{1}{k} \sum_{j=0}^{k-1} \binom{n+1-j}{k+1-j} S_1(n, j), \quad k \geqslant 1,$$

and

(10) $$S_2(n, k) = \frac{1}{k} \sum_{j=0}^{k-1} \binom{-n-j}{k+1-j} S_2(n, j), \quad k \geqslant 1.$$

These are in fact equivalent relations when we use (7) and (8) as intermediaries to pass from the one to the other. Thus a proof of the one yields the other. It is also readily seen that the formula of Mitrinović and Đoković (2) is equivalent to relation (9), for we have only to apply (4) and replace k by $n-k$. Relations (9) and (10) may be obtained separately by use of generating functions, and we remark that this can be done in the context of Nörlund's generalized Bernoulli numbers [10] since these are related to the Stirling numbers by the relations

(11) $$\binom{n-1}{k} B_k^{(n)} = (-1)^k S_1(n-1, k),$$

and

(12) $$\binom{n+k}{k} B_k^{(-n)} = S_2(n, k),$$

where $B_k^{(x)} = B_k^{(x)}(0)$ is a generalized Bernoulli number and $B_k^{(x)}(t)$ is defined by

$$(13) \qquad \left(\frac{z}{e^z - 1}\right)^x e^{tz} = \sum_{k=0}^{\infty} B_k^{(x)}(t) \frac{z^k}{k!}.$$

In view of the relation $\binom{-x}{k} = (-1)^k \binom{x+k-1}{k}$ it is evident why (11) and (12) imply (7) and (8).

We shall examine a simple proof of (9) using finite series only, and see that it suggests a formula of similar nature which appears to be new.

Now clearly (9) is equivalent to the formula

$$(14) \qquad n\, S_1(n-1, k) = \sum_{j=0}^{k} \binom{n-j}{k+1-j} S_1(n-1, j),$$

and by means of formula (1) above

$$(15) \qquad \binom{x}{n} = \sum_{j=0}^{n} S_1(n-1, j) \frac{(-1)^j}{n!} x^{n-j}$$

it is also evident that the left-hand member of (14) is the coefficient of $(-1)^k x^{n-k}/n!$ in the expansion of $n\binom{x}{n}$. To establish that the same is true of the right-hand member of (14) we proceed as follows.
We have

$$\sum_{k=0}^{n} \frac{(-1)^k}{n!} x^{n-k} \sum_{j=0}^{k} \binom{n-j}{k+1-j} S_1(n-1, j)$$

$$= \frac{1}{n!} \sum_{j=0}^{n} S_1(n-1, j) \sum_{k=0}^{n-j} (-1)^{k+j} \binom{n-j}{k+1} x^{n-k-j}$$

$$= \frac{x}{n!} \sum_{j=0}^{n} (-1)^{j-1} S_1(n-1, j) \left\{(x-1)^{n-j} - x^{n-j}\right\}$$

$$= -x\binom{x-1}{n} + x\binom{x}{n} = n\binom{x}{n}.$$

This establishes the result.

The appearance of the term $(x-1)^{n-j} - x^{n-j}$ suggests a variation. It is familiar that

$$(16) \qquad \mathop{\triangle}_{x,\,1} \binom{x}{n} = \binom{x}{n-1} = \sum_{k=0}^{n-1} S_1(n-2, k) \frac{(-1)^k}{(n-1)!} x^{n-1-k},$$

by (15).
However, if we first expand $\binom{x}{n}$ by means of (15) and then calculate the result of applying the difference operator, it is easy to show that we have

$$(17) \qquad \mathop{\triangle}_{x,\,1} \binom{x}{n} = \sum_{k=0}^{n-1} x^{n-k-1} \sum_{j=0}^{k} (-1)^j \binom{n-j}{n-k-1} \frac{1}{n!} S_1(n-1, j).$$

Indeed

$$\underset{x,1}{\triangle}\binom{x}{n}=\sum_{j=0}^{n} S_1(n-1,j)\,\frac{(-1)^j}{n!}\left\{(x+1)^{n-j}-x^{n-j}\right\}$$

$$=\sum_{j=0}^{n-1} S_1(n-1,\,j)\,\frac{(-1)^j}{n!}\left\{(x+1)^{n-j}-x^{n-j}\right\}$$

$$=\sum_{j=0}^{n-1} S_1(n-1,\,j)\,\frac{(-1)^j}{n!}\sum_{k=0}^{n-j-1}\binom{n-j}{k}x^k$$

$$=\sum_{k=0}^{n-1} x^k \sum_{j=0}^{n-k-1}(-1)^j\binom{n-j}{k}\frac{1}{n!}\,S_1(n-1,j)$$

$$=\sum_{k=0}^{n-1} x^{n-1-k}\sum_{j=0}^{k}(-1)^j\binom{n-1}{n-1-k}\frac{1}{n!}\,S_1(n-1,j),$$

which establishes (17).

Upon equating coefficients in (16) and (17) there results the identity

$$\sum_{j=0}^{k}(-1)^j\binom{n-j}{n-k-1}S_1(n-1,\,j)=(-1)^k\,n\,S_1(n-2,\,k),$$

which may be put into the more elegant form

$$\sum_{j=0}^{k}(-1)^j\binom{n-j}{k+1-j}S_1(n-1,\,j)=(-1)^k\,n\,S_1(n-2,\,k). \tag{18}$$

This is an interesting companion to formula (14) in that we have been able to introduce the factor $(-1)^j$. As a consequence we also obtain the two recurrences below:

$$\sum_{j=0}^{[k/2]}\binom{n-2j}{k+1-2j}S_1(n-1,2j)=n\left\{S_1(n-1,k)+(-1)^k S_1(n-2,k)\right\} \tag{19}$$

and

$$\sum_{j=0}^{[(k-1)/2]}\binom{n-2j-1}{k-2j}S_1(n-1,2j+1)=n\left\{S_1(n-1,k)-(-1)^k S_1(n-2,k)\right\}. \tag{20}$$

Of course, similar results follow for the Stirling numbers of the second kind, $S_2(n,k)$, by merely applying (8).

It would be of considerable interest if general higher differences were applied in (16) and (17), instead of just the first difference, however we shall not take up this question here.

The writer wishes to indicate that this is an improved version of a paper originally written in April of 1961, but which paper received only a very limited circulation. Some notational improvements have been made. References $1-5$ give some further results for S_1 and S_2 and in particular an extension to the q-binomial coefficient expansion.

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