PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 6 (20), 1966, p. 105—106.

INEQUALITIES OF R. RADO TYPE FOR WEIGHTED MEANS

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(Communicated November 12, 1965)

1. Let $a=(a_1, a_2, \ldots)$, $p=(p_1, p_2, \ldots)$, $q=(q_1, q_2, \ldots)$ be sequences of positive numbers.

For $n=1, 2, \ldots$ we set

$$A_n(a;\ q)=\sum_1^n a_i q_i \Big/ \sum_1^n q_i,$$

$$G_n(a;\ p)=\left(\prod_1^n a_i^{\ p_i}\right)^{1/\sum\limits_1^n p_i}.$$

We shall prove the following inequality:

$$(1)\qquad \left(\sum_1^n q_i\right) A_n(a;\ q)-\frac{q_n}{p_n}\left(\sum_1^n p_i\right) G_n(a;\ p)$$
$$\geqslant\left(\sum_1^{n-1} q_i\right) A_{n-1}(a;\ q)-\frac{q_n}{p_n}\left(\sum_1^{n-1} p_i\right) G_{n-1}(a;\ p).$$

We put

$$(2)\qquad f(a_n)=\sum_1^n a_i q_i-\frac{q_n}{p_n}\left(\sum_1^n p_i\right)\left(\prod_1^n a_i^{\ p_i}\right)^{1/\sum\limits_1^n p_i}$$
$$=\left(\sum_1^{n-1} q_i\right) A_{n-1}(a; q)+a_n q_n$$
$$-\frac{q_n}{p_n}\left(\sum_1^n p_i\right) G_{n-1}(a;\ p)^{\sum\limits_1^{n-1} p_i/\sum\limits_1^n p_i}\, a_n^{\ p_n/\sum\limits_1^n p_i}.$$

By differentiation we get

$$(3)\qquad f'(a_n)=q_n-q_n\left[\frac{1}{a_n}\, G_{n-1}(a;\ p)\right]^{\sum\limits_1^{n-1} p_i\ \sum\limits_1^n p_i},$$

$$(4)\qquad f''(a_n)=\frac{q_n}{a_n}\cdot\frac{\sum_1^{n-1}p_i}{\sum_1^n p_i}\left[\frac{1}{a_n}\,G_{n-1}(a;p)\right]^{\sum_1^{n-1}p_i/\sum_1^n p_i}>0.$$

From (3) and (4) we conclude that $f(a_n)$ has a minimum for $a_n = G_{n-1}(a;p)$ and no more extrema. From (3) we get

$$\min f(a_n)=\left(\sum_1^{n-1}q_i\right)A_{n-1}(a;q)-\frac{q_n}{p_n}\left(\sum_1^{n-1}p_i\right)\cdot G_{n-1}(a;p),$$

and (1) follows.

2. If $p_k=q_k$ $(k=1, 2, \ldots)$, we get from (1)

$$(5)\qquad \begin{aligned} 0 &\leqslant (p_1+p_2)\,[A_2(a;p)-G_2(a;p)] \\ &\leqslant \cdots \\ &\leqslant \left(\sum_1^{n-1}p_i\right)[A_{n-1}(a;p)-G_{n-1}(a;p)] \\ &\leqslant \left(\sum_1^{n}p_i\right)[A_{n}(a;p)-G_{n}(a;p)] \\ &\leqslant \cdots \end{aligned}$$

If $A_n(a;p)=G_n(a;p)$, we must have

$$A_k(a;p)=G_k(a;p)\quad (k=1, 2, \ldots),\ \text{ i. e.},\ a_1=a_2=\cdots=a_n.$$

3. If $p_1=p_2=\cdots=p_n=1$, (5) gives

$$(6)\qquad n(A_n-G_n)\geqslant(n-1)(A_{n-1}-G_{n-1}),$$

where

$$A_k=\frac{1}{k}\sum_1^k a_i,\quad G_k=\left(\prod_1^k a_i\right)^{1/k}.$$

Inequality (6) is due to R. Rado [1]. The same inequality was rediscovered by L. Tchakaloff [2], E. Jacobsthal [3] and A. Dinghas [4].

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