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SOME POLYNOMIALS RELATED TO THE GENERALIZED LAGUERRE POLYNOMIALS

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1. In a recent paper [1] Carlitz has considered a set of polynomials $A_n^{(\lambda)}(x)$ such that

$$\sum_{r=0}^{n} A_r^{(\lambda)}(x)\, P_{n-r}^{(\lambda+r)}(x) = 0, \quad (n \geqslant 1) \tag{1.1}$$

$$A_0^{(\lambda)}(x) = 1,$$

where $P_n^{(\lambda)}(x)$ is the ultraspherical polynomial of degree n, defined dy

$$(1-2xt+t^2)^{-\lambda} = \sum_{n=0}^{\infty} t^n P_n^{(\lambda)}(x), \tag{1.2}$$

Following the same plan, R. P. Singh [2] has considered a set of polynomials $A_n^{(\alpha)}(x, r, p)$ such that

$$\sum_{k=0}^{n} A_k^{(\alpha)}(x, r, p)\; L_{n-k}^{\alpha+k}(x, \gamma, p) = 0, \quad (n \geqslant 1) \tag{1.3}$$

$$A_0^{(\alpha)}(x, r, p) = 1,$$

where $L_n^{(\alpha)}(x, r, p)$ is the generalized Laguerre polynomials defined by

$$L_n^{(\alpha)}(x, r, p) = \frac{x^{-\alpha} e^{px^r}}{n!} D^n (x^{\alpha+n} e^{-px^r}) \tag{1.4}$$

For the polynomials $A_n^{(\alpha)}(x, r, p)$ we mention the following properties

$$\sum_{k=0}^{\infty} A_k^{(\alpha)}(x, r, p)\, \zeta^k = (1+\zeta)^{-\alpha-1} exp[p x^r]\{(1+\zeta)^r - 1\} \tag{1.5}$$

$$A_n^{(\alpha)}(x,r,p) = L_n^{-\alpha-n-1}(x,r,-p) \tag{1.6}$$

(1.7) $$A_n^{(\alpha)}(x,r,p) = \frac{e^{-px^r}\, x^{\alpha+n+1}}{n!}\, D^n(x^{-\alpha-1}e^{p\,x^r})$$

(1.8) $$\sum_{n=0}^{\infty} \frac{(n+k)!}{n!\;k!}\; A_{n+k}^{(\alpha)}(x,r,p)\,\zeta^n$$

$$=(1+\zeta)^{-1-\alpha-k}\; e^{p\,x^r\,\{(1+\zeta)^r-1\}} \times A_k^{(\alpha)}\{(1+\zeta)\,x,r,p\}\;.$$

Here we like to point out that $L_n^{(\alpha)}(x,r,p)$ will be a polynomial of degree rn, provided r is a natural number. For this reason, one of the authors [3] of the present paper had already used the notation $T_{kn}^{(\alpha)}(x,p)$ defined by

(1.9) $$T_{kn}^{(\alpha)}(x,p) = \frac{1}{n!}\, x^{-\alpha}\, e^{px^k}\, D^n\left(x^{\alpha+n}\, e^{-px^k}\right),$$

and generated by

(1.10) $$\sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x,p)\, t^n = (1-t)^{-\alpha-1}\, exp[p\,x^k\{1-(1-t)^{-k}\}].$$

Throughout this paper, we shall use the notation $T_{kn}^{(\alpha)}(x,p)$ instead of $L_n^{(\alpha)}(x,k,p)$. Also in the definition (1.3) of $A_n^{(\alpha)}(x\;k,\,p)$ we shall use the notation $A_{kn}^{(\alpha)}(x,\,p)$.

The object of this paper is to point ont that a simple comparison of the result [4]

(1.11) $$\sum_{n=0}^{\infty} T_{kn}^{(\alpha-n)}(x,p)\, t^n = (1+t)^{\alpha}\, exp[p\,x^k\{1-(1+t)^k\}]$$

with (1.5) implies

(1.12) $$A_{kn}^{(\alpha)}(x,p) = T_{kn}^{(-\alpha-1-n)}(x,-p)$$

which can be considered as the definition of $A_{kn}^{(\alpha)}(x,p)$

Using this definition, we shall deduce properties of $A_{kn}^{(\alpha)}(x,p)$ some of which are not mentioned in [2]. It is interesting to note that (1.3) i. e. (2.1) is a consequence of the two generating functions (1.10) and (1.11).

2. In terms of our notation, (1.3) is stated as follows

(2.1) $$\sum_{r=0}^{n} A_{kr}^{(\alpha)}(x,p)\, T_{k(n-r)}^{(\alpha+r)}(x,p) = 0,\;\; (n \geqslant 1)$$

$$A_0^{(\alpha)}(x,p) \qquad = 1.$$

we shall first show that $A_{kr}^{(\alpha)}(x,p) = T_{kr\,(x,-p)}^{(-\alpha-1-r)}$ satisfies (2.1). For this, we mention two generating functions of Chatterjee [3], [4].

(2.2) $$\sum_{n=0}^{\infty} T_{kn}^{(\alpha)}(x,p)\, t^n = (1-t)^{-\alpha-1}\, exp[p\,x^k\{1-(1-t)^{-k}\}]$$

(2.3) $$\sum_{n=0}^{\infty} T_{kn}^{(\alpha-n)}(x,p)\, t^n = (1+t)^{\alpha}\, exp[p\,x^k\{1-(1+t)^k\}]$$

Now we shall prove that

$$1=\sum_{n=0}^{\infty} t^n \sum_{r=0}^{n} T_{kr}^{-\alpha-1-r}(x,-p)\, T_{k(n-r)}^{\alpha+r}(x,p).$$

For, the right member is equal to

$$\sum_{r=0}^{\infty} t^r\, T_{kr}^{-\alpha-1-r}(x,-p) \sum_{n=0}^{\infty} t^n\, T_{kn}^{\alpha+r}(x,p)$$

$$=\sum_{r=0}^{\infty} t^r\, T_{kr}^{-\alpha-1-r}(x,-p)\cdot(1-t)^{-\alpha-1-r}\, exp[p x^k\{1-(1-t)^{-k}\}]$$

$$=(1-t)^{-\alpha-1} exp[p x^k\{1-(1-t)^{-k}\}] \sum_{r=0}^{\infty}\left(\frac{t}{t-1}\right) T_{kr}^{(-\alpha-1)-r}(x,-p)$$

which stands thus

$$(1-t)^{-\alpha-1} exp[p x^k\{1-(1-t)^{-k}\}]\left(1+\frac{t}{1-t}\right)^{-\alpha-1} exp\left[-p x^k\left\{1-\left(1+\frac{t}{1-t}\right)^k\right\}\right.$$

$$=(1-t)^{-\alpha-1} e^{px^k}\ {}^{\{1-(1-t)^{-k}\}}.\,(1-t)^{\alpha+1}\, e^{-px^k}\ {}^{\{1-(1-t)^{-k}\}}=1$$

Moreover a simple comparison of (1.5) and (2.3) implies at once

$$A_{kn}^{(\alpha)}(x,p)=T_{kn}^{(-\alpha-1-n)}(x,-p). \tag{2.4}$$

Again for the generating function of $A_{kn}^{(\alpha)}(x,p)$, we may notice that

$$\sum_{n=0}^{\infty} A_{kn}^{(\alpha)}(x,\ p)\ t^n=\sum_{n=0}^{\infty} T_{kn}^{(-\alpha-1-n)}(x,-p)\, t^n$$

$$=\sum_{n=0}^{\infty}\frac{t^n}{n!} e^{-px^k}\ x^{\alpha+1+n} D^n\left(x^{-\alpha-1} e^{px^k}\right)$$

$$=x^{\alpha+1} exp[-px^k] \sum_{n=0}^{\infty}\frac{(x\,t)^n}{n\,!} D^n\left(x^{-\alpha-1}\ e^{px^k}\right)$$

Now by Taylor's formula we know

$$\sum_{n=0}^{\infty}\frac{(x\,t)^n}{n\,!} D^n f(x)=f(x(1+t)).$$

It follows therefore

$$\sum_{n=0}^{\infty} A_{nk}^{(x)}(x,\ p)\ t^n=x^{\alpha+1} e^{-px^k}\ \{x(1+t)\}^{-\alpha-1} e^{p\,x^k\,(1+t)^k} \tag{2.5}$$

$$=(1+t)^{-\alpha-1} e^{p\,x^k}\ \{(1+t)^k-1\},$$

which is evidently (1.5).

In this connection, we may prove the following generating functions:

(2.6)
$$\sum_{n=0}^{\infty} A_{kn}^{(\alpha-n)}(x, p)(xt)^n = (1-t)^{\alpha} e^{px^k\{(1-t)^{-k}-1\}}$$

(2.7)
$$\sum_{n=0}^{\infty} A_{kn}^{(\alpha-2n)}(x, p)(xt)^n =$$

$$= (1-4xt)^{-\frac{1}{2}} \left(\frac{1-\sqrt{1-4xt}}{2xt}\right)^{-\alpha-1} e^{p\left\{\left(\frac{1-\sqrt{1-4xt}}{2t}\right)^k - x^k\right\}}$$

It may be noted that (2.6) follows easily from (2.4) and (2.2). Next to prove (2.7) we require Lagrange's expansion:

(2.8)
$$\frac{F(\zeta)}{1-t\,\Phi'(\zeta)} = \sum \frac{t^n}{n!} D^n \{(\Phi(x))^n \cdot F(x)\}$$

where
$$\zeta = x + t\,\Phi(\zeta)$$
Now

(2.9)
$$\sum_{n=0}^{\infty} A_{kn}^{(\alpha-2n)}(x, p) \cdot (xt)^n$$

$$= x^{\alpha+1} e^{-px^k} \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n \{x^{2n} \cdot x^{-\alpha-1} \cdot e^{px^k}\}$$

In (2.8), let $\Phi(x) = x^2$, $F(x) = x^{-\alpha-1} e^{px^k}$, thus we have

$$\zeta = x + t\zeta^2 \qquad \left(|t| < \frac{1}{4x}\right)$$

so that $\zeta = (1-\sqrt{1-4xt})/2t$, because this root tends to x as $t \to \infty$. It therefore follows from (2.8) and (2.9)

$$\sum_{n=0}^{\infty} A^{(\alpha-2n)}(x, p)(xt)^n$$

$$= (1-4xt)^{-\frac{1}{2}} \left(\frac{1-\sqrt{1-4xt}}{2xt}\right)^{-\alpha-1} e^{p\left\{\left(\frac{1-\sqrt{1-4xt}}{2t}\right)^k - x^k\right\}}$$

$$\left(|t| < \frac{1}{4x}\right)$$

In [3], we notice that

(2.10)
$$\prod_{j=1}^{n} (x D - pk x^k + \alpha - j) Y$$
$$= n! \sum_{r=0}^{n} \frac{x^r}{r!} T_{k(n-r)}^{(\alpha+r)} (x, p) D^r Y$$

where Y is any sufficiently differentiable function of x. Changing α to $-\alpha-1-n$ and p to $-p$, we derive

$$\prod_{j=1}^{n} (x D + pk x^k - \alpha - 1 - n + j) Y$$
$$= n! \sum_{r=0}^{n} \frac{x^r}{r!} T_{k(n-r}^{(-\alpha-1-n+r)} (x, -p) D^r Y$$
$$= n! \sum_{r=0}^{n} \frac{x^r}{r!} A_{k(n-r)}^{(\alpha)} (x, p) D^r Y.$$

In other words,

(2.11)
$$\prod_{i=0}^{n-1} (x D + pk x^k - \alpha - 1 - i) Y$$
$$= n! \sum_{r=0}^{n} \frac{x^r}{r!} A_{k(n-r)}^{(\alpha)} (x, p) D^r Y.$$

In particular we have

(2.12)
$$n!\ A_{kn}^{(\alpha)} (x, p) = \prod_{i=0}^{n-1} (x D + pk x^k - \alpha - 1 - i) \cdot 1$$

Now we note some consequences of (2.12). We observe

$$(n+1)!\, A_{k(n+1)}^{(\alpha)} (x, p) = (x D + pk x^k - \alpha - 1 - n) \prod_{i=0}^{n-1} (x D + pk x^k - \alpha - 1 - i)\, 1$$
$$= (x D + pk x^k - \alpha - 1 - n)\, n!\ A_{kn}^{(\alpha)} (x, p)$$

whence we obtain

(2.13)
$$(n+1)\, A_{k(n+1)}^{(\alpha)} (x, p) = (x D + pk x^k - \alpha - 1 - n)\, A_{kn}^{(\alpha)} (x, p),$$

a formula which Singh [2] has derived in a different manner.

Next we consider

$$(m+n)!\ A_{k(m+n)}^{(\alpha)}(x,\ p)$$

$$= \prod_{j=0}^{m-1} (x\,D + pk\,x^k - \alpha - n - 1 - j) \prod_{i=0}^{n-1} (x\,D + pk\,x^k - \alpha - 1 - i) \cdot 1$$

$$= \prod_{j=0}^{m-1} (x\,D + pk\,x^k - \alpha - n - 1 - j)\,n!\ A_{kn}^{(\alpha)}(x,\ p)$$

$$= n!\ m! \sum_{r=0}^{m} \frac{x^r}{r!} A_{k(m-r)}^{(\alpha-n)}(x,\ p)\,D^r\,A_{kn}^{(\alpha)}(x,\ p)$$

which implies that

$$\binom{m+n}{m} A_{k(m+n)}^{(\alpha)}(x,\ p) = \sum_{r=0}^{\min(m,\ n)} \frac{x^r}{r!} A_{k(m-r)}^{\alpha-n}(x,\ p)\,D^r\,A_{kn}^{(\alpha)}(x,\ p)$$

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