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ON CERTAIN TRANSFORMATIONS IN OPERATIONAL CALCULUS

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1. Introduction. Mainra [3] has defined

$$\tilde{\omega}^{\nu}_{\mu,\lambda}(x)=\int_0^{\infty}\tilde{\omega}_{\mu,\lambda}(xy)\,J_{\nu}(y)\sqrt{y}\,dy,\ R(\mu,\lambda,\nu)\geqslant -\frac{1}{2}; \tag{1.1}$$

and has established that $\tilde{\omega}^{\nu}_{\mu,\lambda}(x)$ is a Fourier kernel and plays the role of a transform.

Bhatnagar [1] has defined

$$\tilde{\omega}_{\mu,\lambda,\nu}(x)=\int_0^{\infty}\tilde{\omega}_{\mu,\lambda}(x/t)\,J_{\nu}(t)\frac{dt}{\sqrt{t}},\ (\mu,\lambda,\nu)\geqslant -\frac{1}{2}; \tag{1.2}$$

and proved that this is also a Fourier kernel and plays the role of a transform.

Further the integral equation

$$\psi(p)=p\int_0^{\infty}e^{-pt}f(t)\,dt,\ R(p)>0; \tag{1.3}$$

is symbolically denoted as

$$\psi(p)\doteqdot f(t) \text{ or } f(t)\doteqdot\psi(p),$$

where $\psi(p)$ is known as the classical Laplace transform of $f(t)$.

In this paper, we shall establish that if $f(t)\doteqdot\psi(p)$, $g(t)\doteqdot\phi(p)$ and the two originals $f(t)$, $g(t)$ are related by any of the Fourier kernels defined above, then their images $\psi(p)$, $\phi(p)$ are also related by one of the Fourier kernels different from the first one and vice — versa. We may add that converse theorems can be proved either under similar conditions or under modified conditions. As $\tilde{\omega}^{\nu}_{\mu,\lambda}(x)$ and $\tilde{\omega}_{\mu,\lambda,\nu}(x)$ involve Bessel functions, it is well-known that $R(\mu,\lambda,\nu)>-1$. So at some places in the conditions we have not mentioned it as this is implied. For self-reciprocity we take $R(\mu,\lambda,\nu)\geqslant -\frac{1}{2}$.

2. Let
$$f(t)\doteqdot\psi(p) \tag{2.1}$$
and
$$F(at)\doteqdot\varkappa(p/a). \tag{2.2}$$

Applying Goldstein's theorem [2] to (2.1) and (2.2), we get

$$\int_0^\infty f(x)\, \varkappa\,(x/a)\,\frac{dx}{x}=\int_0^\infty \psi(x)\,F(ax)\,\frac{dx}{x}, \tag{2.3}$$

provided the necessary change in the order of integrations involved are permissible and the integrals converge.

Putting $a=\dfrac{1}{p}$ and interpreting with the help of (2.2), we get

$$\int_0^\infty F(y/x)\,f(x)\,\frac{dx}{x} \doteqdot \int_0^\infty F(x/p)\,\psi(x)\,\frac{dx}{x}. \tag{2.4}$$

Let us put $F(t)=t^m\,\tilde{\omega}_{\mu,\,\lambda}(1/t)$ in (2.4). We get

$$y^m \int_0^\infty \tilde{\omega}_{\mu,\,\lambda}(x/y)\,f(x)\,x^{-1-m}\,dx \doteqdot p^{-m}\int_0^\infty \tilde{\omega}_{\mu,\,\lambda}(p/x)\,\psi(x)\,x^{m-1}\,dx. \tag{2.5}$$

Now Mitra and Bose [4] have proved that if $f_1(t)\doteqdot\psi_1(p)$,

$$\text{then } t^{\nu+1}\int_0^\infty J_\nu(tz)\,f_1(z)\,z^{-\nu}dz \doteqdot p^{1-\nu}\int_0^\infty J_{\nu+1}(pz)\,\psi_1(z)\,z^\nu\,dz, \tag{2.6}$$

$$R(\nu)\geqslant -\frac{1}{2}.$$

Making use of (2.5) in (2.6), we get

$$t^{\nu+1}\int_0^\infty J_\nu(tz)\,z^{m-\nu}\,dz\int_0^\infty \tilde{\omega}_{\mu,\,\lambda}(x/z)\,f(x)\,x^{-1-m}\,dx$$

$$\doteqdot p^{1-\nu}\int_0^\infty J_{\nu+1}(pz)\,z^{\nu-m}\,dz\int_0^\infty \tilde{\omega}_{\mu,\,\lambda}(z/x)\,\psi(x)\,x^{m-1}\,dx.$$

Changing the order of integrations on both the sides and putting

$$m=\nu-\frac{1}{2},\text{ we get}$$

$$t^{\nu+1}\int_0^\infty f(x)\,x^{-\frac{1}{2}-\nu}\,dx\int_0^\infty \tilde{\omega}_{\mu,\,\lambda}(x/z)\,J_\nu(tz)\,z^{-\frac{1}{2}}\,dz$$

$$\doteqdot p^{1-\nu}\int_0^\infty \psi(x)\,x^{\nu-\frac{3}{2}}\,dx\int_0^\infty \tilde{\omega}_{\mu,\,\lambda}(z/x)\,J_{\nu+1}(pz)\,\sqrt{z}\,dz,$$

provided $x^{\nu-\frac{1}{2}}\psi(x)$ and $x^{-\nu}f(x)$ are bounded and absolutely integrable in $(0,\infty)$ and $R\left(\mu,\nu,\lambda,+\frac{1}{2}\right)>0$.

On writing z/t for z on the l. h. s. and z/p for z on the r. h. s. and making use of the definitions of $\tilde{\omega}_{\mu,\,\lambda\,\nu,}(x)$ and $\tilde{\omega}^{\nu}_{\mu,\,\lambda}(x)$, the above can be written after a little change as

$$t^{\nu+\frac{1}{2}}\int_0^\infty \tilde{\omega}_{\mu,\,\lambda,\,\nu}(tx)\,f(x)\,x^{-\frac{1}{2}-\nu}\,dx \quad p^{-\nu-\frac{1}{2}}\int_0^\infty \tilde{\omega}^{\nu+1}_{\mu,\,\lambda}(x/p)\,\psi(1/x)\,x^{-\frac{1}{2}-\nu}dx. \tag{2.7}$$

Let us put $\phi(p)=p^{-\nu-\frac{1}{2}}\int_0^\infty \tilde{\omega}_{\mu,\lambda}^{\nu+1}(x/p)\,\psi(1/x)\,x^{-\frac{1}{2}-\nu}\,dx=$ r. h. s. of (2.7)

i. e. $p^{-\nu-\frac{1}{2}}\phi(1/p)$ is the $\tilde{\omega}_{\mu,\lambda}^{\nu+1}(x)$ transform of $p^{-\nu-\frac{1}{2}}\psi(1/p)$.

Taking $g(t)\doteqdot\phi(p)$, we get

$$g(t)=t^{\nu+\frac{1}{2}}\int_0^\infty \tilde{\omega}_{\mu,\lambda,\nu}(tx)\,f(x)\,x^{-\frac{1}{2}-\nu}\,dx=\text{l.h.s. of (2.7)},$$

which shows that $t^{-\nu-\frac{1}{2}}g(t)$ is the $\tilde{\omega}_{\mu,\lambda,\nu}(x)$ transform of $t^{-\nu-\frac{1}{2}}f(t)$. Thus:

Theorem 1. Let $f(t)\doteqdot\psi(p)$, $g(t)\doteqdot\phi(p)$ and $p^{-\nu-\frac{1}{2}}\phi(1/p)$ be the $\tilde{\omega}_{\mu,\lambda}^{\nu+1}(x)$ transform of $p^{-\nu-\frac{1}{2}}\psi(1/p)$.

Then $t^{-\nu-\frac{1}{2}}g(t)$ will be the $\tilde{\omega}_{\mu,\lambda,\nu}(x)$ transform of $t^{-\nu-\frac{1}{2}}f(t)$, provided $x^{\nu-\frac{1}{2}}\psi(x)$ and $x^{-\nu}f(x)$ are bounded and absolutely integrable in $(0,\infty)$, $g(t)$ and $t^{\nu+\frac{1}{2}}\int_0^\infty \tilde{\omega}_{\mu,\lambda,\nu}(tx)f(x)\,x^{-\frac{1}{2}-\nu}\,dx$ are continuous functions of t in $(0,t)$ and $R\left(\mu,\lambda,\nu,+\frac{1}{2}\right)>0$.

Further let $x^{-\nu-\frac{1}{2}}\psi(1/x)$ be $R_{\mu,\lambda}^{\nu+1}$; then from (2.7), we have

$$t^{\nu+\frac{1}{2}}\int_0^\infty \tilde{\omega}_{\mu,\lambda,\nu}(tx)\,f(x)\,x^{-\frac{1}{2}-\nu}\,dx\doteqdot\psi(p).\ \text{But}\ \psi(p)\doteqdot f(t).$$

$$\text{(2.8)}\qquad \therefore\quad f(t)=t^{\nu+\frac{1}{2}}\int_0^\infty \tilde{\omega}_{\mu,\lambda,\nu}(tx)\,f(x)\,x^{-\frac{1}{2}-\nu}\,dx,$$

which shows that $x^{-\nu-\frac{1}{2}}f(x)$ is $R_{\mu,\lambda,\nu}$, provided both the sides of (2.8) are continuous functions of t.

Thus:

Cor. 1. Let $f(t)\doteqdot\psi(p)$ and $x^{-\nu-\frac{1}{2}}\psi(1/x)$ be $R_{\mu,\lambda}^{\nu+1}$. Then $x^{-\nu-\frac{1}{2}}f(x)$ will be $R_{\mu,\lambda,\nu}$, provided the conditions of the theorem are satisfied and $\left[f(t)-t^{\nu+\frac{1}{2}}\int_0^\infty \tilde{\omega}_{\mu,\lambda,\nu}(tx)\,f(x)\,x^{-\frac{1}{2}-\nu}\,dx\right]$ is a continuous function of t.

Again putting $\lambda=\nu+1$ in (2.7), we can write it after changing the sides as

$$\text{(2.9)}\qquad p^{-\nu-\frac{1}{2}}\int_0^\infty J_\mu(x/p)\,\sqrt{x/p}\,\psi(1/x)\,x^{-\frac{1}{2}-\nu}\,dx$$

$$=p\int_0^\infty e^{-pt}\,t^{\nu+\frac{1}{2}}\,dt\int_0^\infty \tilde{\omega}_{\mu,\nu+1,\nu}(tx)\,f(x)\,x^{-\frac{1}{2}-\nu}\,dx$$

$$=\int_0^\infty \tilde{\omega}_{\mu,\nu+1,\nu}(x)\,x^{-\nu-\frac{1}{2}}\left[p\int_0^\infty e^{-pt}f(x/t)\,t^{2\nu}\,dt\right]dx$$

$$=\int_0^\infty \tilde{\omega}_{\mu,\nu+1,\nu}(x)\,x^{\nu-\frac{1}{2}}\left[px\int_0^\infty e^{-pxt}f(1/t)\,t^{2\nu}\,dt\right]dx$$

$$= \int_0^\infty \tilde{\omega}_{\mu,\,\nu+1,\,\nu}(x)\, F(px)\, x^{\nu-\frac{1}{2}}\, dx, \text{ [assuming } F(p) \doteq t^{2\nu} f(1/t)].$$

Let $\xi(p)$ be the Hankel-transform of order μ of $x^{-\nu-\frac{1}{2}}\psi\left(\frac{1}{x}\right)$.

Then from (2.9), we get

$$p^{-\nu-\frac{1}{2}}\xi(1/p) = \int_0^\infty \tilde{\omega}_{\mu,\,\nu+1,\,\nu}(x)\, F(px)\, x^{\nu-\frac{1}{2}}\, dx$$

or

$$\xi(1/p) = \int_0^\infty \tilde{\omega}_{\mu,\,\nu+1,\,\nu}(x/p)\, F(x)\, x^{\nu-\frac{1}{2}}\, dx,$$

which shows that $\xi(p)$ is the $\tilde{\omega}_{\mu,\,\nu+1,\,\nu}(x)$ transform of $p^{\nu-\frac{1}{2}}F(p)$. Thus:

Cor. 2. Let $f(t) \doteq \psi(p)$, $t^{2\nu} f(1/t) \doteq F(p)$ and $\xi(p)$ be the Hankel-transform of order μ of $x^{-\nu-\frac{1}{2}}\psi(1/x)$.

Then $\xi(p)$ will be the $\tilde{\omega}_{\mu,\,\nu+1,\,\nu}(x)$ transform of $p^{\nu-\frac{1}{2}}F(p)$, provided the conditions of the theorem hold good.

Further, if we put $t^{2\nu} f(1/t) = f(t)$ in the above corollary, we get $\psi(p) = F(p)$. Hence we have

Cor. 3. Let $f(t) = t^{2\nu} f(1/t)$, $f(t) \doteq \psi(p)$ and $\xi(p)$ be the Hankel-transform of order μ of $x^{-\nu-\frac{1}{2}}\psi(1/x)$.

Then $\xi(p)$ will be the $\tilde{\omega}_{\mu,\,\nu+1,\,\nu}(x)$ transform of $p^{\nu-\frac{1}{2}}\psi(p)$ provided the conditions of the theorem hold good.

Again, we know that if $f(t) \doteq \psi(p)$,

(2.10) then $$\left(t\frac{d}{dt}\right)^n f(t) \doteq (-1)^n \left(p\frac{d}{dp}\right)^n \psi(p).$$

Making use of the above relation in (2.7), we get

(2.11) $$t^{\nu+\frac{1}{2}}\int_0^\infty \tilde{\omega}_{\mu,\,\lambda,\,\nu}(tx)\left[\left(x\frac{d}{dx}\right)^n f(x)\right] x^{-\nu-\frac{1}{2}}\, dx \doteq p^{-\nu-\frac{1}{2}}\int_0^\infty \tilde{\omega}^{\nu+1}_{\mu,\,\lambda}(x/p)\times$$
$$\times\left[(-1)^n\left(\frac{1}{x}\frac{d}{d\frac{1}{x}}\right)^n \psi(1/x)\right] x^{-\nu-\frac{1}{2}}\, dx,$$

provided $x^{-\nu-\frac{1}{2}}\left(x\frac{d}{dx}\right)^n f(x)$ and $x^{-\nu-\frac{1}{2}}\left(\frac{1}{x}\frac{d}{d\frac{1}{x}}\right)^n \psi(1/x)$ are bounded and absolutely integrable in $(0, \infty)$.

Now proceeding in the same manner as in theorem 1, we have the following theorem:

Theorem 2. Let $f(t) \doteq \psi(p)$, $g(t) \doteq \phi(p)$ and $p^{-\nu-\frac{1}{2}}\phi(1/p)$ be the $\tilde{\omega}^{\nu+1}_{\mu,\,\lambda}(x)$ transform of $(-1)^n p^{-\nu-\frac{1}{2}}\left(\frac{1}{p}\frac{d}{d\frac{1}{p}}\right)^n \psi(1/p)$.

Then $t^{-\nu-\frac{1}{2}}g(t)$ will be the $\tilde{\omega}_{\mu,\lambda,\nu}(x)$ transform of

$$t^{-\nu-\frac{1}{2}}\left(t\frac{d}{dt}\right)^n f(t), \text{ provided } x^{-\nu-\frac{1}{2}}\left(x\frac{d}{dx}\right)^n f(x) \text{ or } x^{-\nu}\left(x\frac{d}{dx}\right)^n f(x)$$

and $x^{-\nu-\frac{1}{2}}\left(\frac{1}{x}\frac{d}{d\frac{1}{x}}\right)^n \psi(1/x)$ are bounded and absolutely integrable in $(0, \infty)$.

Cor. Let $f(t) \doteqdot \psi(p)$ and $(-1)^n x^{-\nu-\frac{1}{2}}\left(\frac{1}{x}\frac{d}{d\frac{1}{x}}\right)^n \psi(1/x)$ be $R^{\nu+1}_{\mu,\lambda}$.

Then $t^{-\nu-\frac{1}{2}}\left(t\frac{d}{dt}\right)^n f(t)$ will be $R_{\mu,\lambda,\nu}$, provided the conditions of the above theorem hold good.

Similarly,

Theorem 3. Let $f(t) \doteqdot \psi(p)$, $g(t) \doteqdot \phi(p)$ and $p^{-\nu-\frac{1}{2}}\phi(1/p)$ be the $\tilde{\omega}^{\nu+1}_{\mu,\lambda}(x)$ transform of $p^{-\nu-n-\frac{1}{2}}\psi(1/p)$.

Then $t^{-\nu-\frac{1}{2}}g(t)$ will be the $\tilde{\omega}_{\mu,\lambda,\nu}(x)$ transform of $t^{-\nu-\frac{1}{2}}f^n(t)$, provided $f^r(0)=0$, $r=0,1,\ldots, n-1$; $x^{-\nu}f^n(x)$ and $x^{-\nu-n}\psi(1/x)$ are bounded and absolutely integrable in $(0, \infty)$.

Cor. Let $f(t) \doteqdot \psi(p)$, $f^r(0)=0$, $r=0,1,2,\ldots, n-1$, and $x^{-\nu-n-\frac{1}{2}}\psi(1/x)$ be $R^{\nu+1}_{\mu,\lambda}$. Then $t^{-\nu-\frac{1}{2}}f^n(t)$ will be $R_{\mu,\lambda,\nu}$, provided the conditions of the above theorem hold good.

Here we make use of the following relation in place of (2.10), i.e. If $f(t) \doteqdot \psi(p)$, then $f^n(t) \doteqdot p^n\psi(p)$, provided $f^r(0)=0$, $r=0,1,2,\ldots, n-1$, and proceed in the same way as in theorem 2.

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