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ON SOME FIRST INTEGRALS OF EQUATIONS OF MOTION

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1. „Similar" Integrals for Spaces in Conformal Correspondence

The usual procedures in analytical dynamics for determination of first integrals of the equations of motion under given conditions produce directly four types of first integrals, the integrals of momentum, the integrals of moment of momentum and the cyclic integrals, as integrals linear in termes of velocity, and, finally, the integral of energy, as an integral of the second order in terms of velocities, which exists in the case of a conservative system. By geometrization of Dynamics, possibilities are created to find, under certain conditions, also first integrals which do not need be only of the types listed above, but the existence of which depends on the structure of the space in which geometrization of a dynamic system is effected. Thus, the integrals the existence of, which, under specific conditions, will be determined below, are indeed much·broader with respect to their importance than those belonging to the class of cyclic integrals.

It is well known that if a Riemannian space V_n admits the solution $\Phi = \Phi(x^1, \ldots\ldots, x^n; C_1, \ldots, C_n)$ of the partial differential equation

$$g^{ij} \frac{\partial \Phi}{\partial x^i} \cdot \frac{\partial \Phi}{\partial x^j} = 1, \qquad (\mathrm{i}, \mathrm{j} = 1,2 \ldots, \mathrm{n}) \tag{1.1}$$

the differential equations of geodesics of the space V_n admit the complete system of n first integrals, linear in terms of generalized velocities, these integrals being of the form

$$\frac{dx^i}{ds} = g^{ij} \frac{\partial \Phi}{\partial x^j} \cdot \qquad (\mathrm{i}, \mathrm{j} = 1,2 \ldots, \mathrm{n}) \tag{1.2}$$

From (1.2), we directly obtain

$$\Phi = s = \int_{No}^{N} \sqrt{g_{ij}\, dx^i\, dx^j}. \tag{1.3}$$

From (1.3) follows a geometrical interpretation of the total integral Φ of the partial equation (1.1). We see that this total integral represents the length of the geodesic in the space considered.

The existence of integrals of the form (1.2) related to the existence of a total integral of the partial equation (1.1) is, in fact, a modern formulation of H. Hertz's results [1]*. Hertz has shown that the trajectory of a system moving under no external forces is a curve of least curvature and that the differential equations of that curve can be written as differential equations of the first order, provided that a total integral of a partial equation determined by the coefficients of kinetic energy is known.

Consider now two Riemannian spaces in conformal correspondence, one of which is assumed to admit first integrals of the linear form (1.2). The problem we consider is whether and under which conditions the existence of „similar" integrals in the other space is possible, so that by means of the known integrals for one space we can write them directly and without any integration for the other space.

Let Φ be a total integral of the partial equation

$$(1.4) \qquad a^{ij}\frac{\partial\Phi}{\partial x^i}\cdot\frac{\partial\Phi}{\partial x^j}=1, \qquad (i,j=1,2\ldots,n)$$

the problem is whether the equation

$$(1.5) \qquad a^{ij}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^j}=\lambda^2(x^1,\ldots,x^n)$$

admits a solution of the form $\psi=\psi(x^1,\ldots,x^n;C_1,\ldots,C_n)$. If (1.5) admits the solution $\psi=\psi(x)$, then ψ must satisfy the condition

$$a^{ij}\left(\frac{1}{\lambda}\frac{\partial\psi}{\partial x^i}\right)\left(\frac{1}{\lambda}\frac{\partial\psi}{\partial x^j}\right)=1,$$

which reduces to

$$(1.6) \qquad \frac{1}{\lambda}\frac{\partial\psi}{\partial x^i}=\frac{\partial\Phi}{\partial x^i}.$$

Therefore the existence of a function ψ, which satisfies (1.5), depends upon the integrability of (1.6) or of the equation

$$(1.7) \qquad \frac{\partial\psi}{\partial x^i}=\lambda\frac{\partial\Phi}{\partial x^i}$$

where Φ is a known solution of the partial differential equation (1.4) and λ is a given scalar function.

The conditions of integrability of (1.7) are

$$\frac{\partial}{\partial x^j}\frac{\partial\psi}{\partial x^j}-\frac{\partial}{\partial x^i}\frac{\partial\psi}{\partial x^j}=0$$

which reduce to

$$(1.8) \qquad \frac{\partial\lambda}{\partial x^j}\frac{\partial\Phi}{\partial x^i}-\frac{\partial\lambda}{\partial x^i}\frac{\partial\Phi}{\partial x^i}=0.$$

If the conditions (1.8) are satisfied, integrating (1.7) we can find the function ψ which will represent a total integral of the partial differential equation (1.5). Hence, we have theorem:

* Numbers in square brackets refer to the references at the end of the paper.

If a Riemannian space V_n with the fundamental form

$$ds^2 = a_{ij}\, dx^i\, dx^j, \qquad (\mathrm{i,j} = 1,2, \ldots, \mathrm{n})$$

admits n first linear integrals of the differential equations of geodesics of the form

$$\frac{dx^i}{ds} = a^{ij} \frac{\partial \Phi}{\partial x^j}$$

then the space $\bar{V}_n$ conformal to the space V_n and with the fundamental form

$$d\bar{s}^2 = \lambda^2\, ds^2 = g_{ij}\, dx^i\, dx^j, \qquad g_{ij} = \lambda^2\, a_{ij}$$

admits n first linear integrals of differential equations of geodesics of the form

$$\frac{dx^i}{d\bar{s}} = \lambda^2 a^{ij} \frac{\partial \Phi}{\partial x^j} \qquad (\mathrm{i,j} = 1,2, \ldots, \mathrm{n})$$

if

$$\frac{\partial \lambda}{\partial x^j} \frac{\partial \Phi}{\partial x^i} - \frac{\partial \lambda}{\partial x^i} \frac{\partial \Phi}{\partial x^j} = 0.$$

Application of these results to the construction of integrals of the equations of motion of scleronomic conservative system is obvious.

2. Generalization of Eisenhart's Theorem on First Integrals of Equations of Geodesics

According to Eisenhart's theorem ([2], p. 129) differential equations of geodesics in a Riemannian space $\bar{V}_n$ will admit integrals of the form

$$\xi^i_{(\lambda)} \frac{dx_i}{d\bar{s}} = \text{const.}, \tag{2.1}$$

when the vectors $\xi_{(\lambda)i}$ satisfy Killing's equations

$$\nabla_i\, \xi_j + \nabla_j\, \xi_i = 0. \tag{2.2}$$

Thus, the existence of integrals of form (2.1), which are linear in terms of first derivatives, is related to the existence of a group of isometric transformations in $\bar{V}_n$. Since Killing's equations (2.2), admit in general, a certain number, say r, of independent solutions

$$\xi_{(\lambda)i}, \qquad \lambda = 1,2, \ldots\ldots, r > 0$$

it follows that if the space $\bar{V}_n$ admits an r-parameter group of isometries it admits also r first integrals of the differential equations of geodesics of the form (2.1).

An account on the existence of integrals of equations of motion of scleronomic conservative systems related to the fundamental vectors $\xi_{(\lambda)i}$ of the group of isometries of the configuration space Vn is given by R. Stojanović [3], [4], in connection with the new formulation of the law of moment of momentum in a form which is independent of the concept of position vector. Since the equations of motion of a scleronomic non-conservative system can be written in a Riemannian space V_n in the form

$$\frac{\delta v^i}{dt} = Q^i$$

where Q^i are components of the generalized force, it has been shown that if a Riemannian space V_n admits a group Gr of isometries with fundamental vectors $\xi_{(\lambda)i}$, then there exists also r — scalar equations

$$\frac{d}{dt}(v^i \xi_{(\lambda)i}) = Q^i \xi_{(\lambda)i} \tag{2.3}$$

which correspond to the given motion. If the generalized forces Q^i are orthogonal to some of the fundamental vectors $\xi_{(\lambda)i}$ of the group Gr of isometries of the configuration space Vn then to each of the vectors corresponds one first linearly independent integral of the equations of motion of the form

$$v^i \xi_{(\lambda)i} = C_{(\lambda)}, \qquad \lambda = 1,2, \dots, r > 0. \tag{2.4}$$

In a generalized affine geodesic space, with the coefficients of linear connection given ([6] 2.29 p. 9) by the relations

$$(A) \quad \Lambda^{\sigma}_{\mu\nu} = \left\{ {\sigma \atop \mu\nu} \right\}_g + \frac{1}{2T}\left(Q_\mu \delta^\sigma_\nu + Q_\nu \delta^\sigma_\mu - Q^\sigma g_{\mu\nu}\right); \ \overline{\Lambda}^{\sigma}_{\mu\nu} = \left\{ {\sigma \atop \mu\nu} \right\}_g - \frac{1}{2T} Q^\sigma g_{\mu\nu}.$$

The differential equations

$$\overset{\overline{\Lambda}}{\nabla}_i \xi_j + \overset{\overline{\Lambda}}{\nabla}_j \xi_i = 0 \tag{2.5}$$

are called quasi-Killing equations, and the vectors $\xi_{(\lambda)i}$ which satisfy such equations are the quasi-Killing vectors ([5], p. 86).

In the geometrization of a special class of rheonomic systems [6] it is shown that the geometrization of scleronomic non-conservative systems can be obtained from it as a special case. Therefore, in a space with coefficients of connection determined by (A) the equations of motion of a non-conservative scleronomic system can be written in the form ([7], p. 8)

$$\frac{\overset{\overline{\Lambda}}{\delta} v^i}{dt} = 0, \quad v^i = \frac{dx^i}{dt}. \tag{2.6}$$

Let us show now that the basis of (2.5) and (2.6), it is possible to generalize the mentioned Eisenhart's theorem for the case of non-Riemannian spaces, and nonconservative systems, as well.

Multiplying (2.6) by the quasi-Killing vectors $\xi_{(\lambda)i}$, we have $\xi_{(\lambda)i} \dfrac{\overset{\overline{\Lambda}}{\delta} v^i}{dt} = 0$

this can be further re-written in the form

$$\xi_{(\lambda)i} \frac{\overset{\overline{\Lambda}}{\delta} v^i}{dt} = \frac{\overset{\overline{\Lambda}}{\delta}}{dt}(\xi_{(\lambda)i} v^i) - v^i \frac{\overset{\overline{\Lambda}}{\delta}}{\delta t} \xi_{(\lambda)i} = 0 \tag{2.7}$$

Since

$$\frac{\overset{\overline{\Lambda}}{\delta}}{\delta t}\xi_{(\lambda)i} = \overset{\overline{\Lambda}}{\nabla}_j \xi_{(\lambda)i}{}^j v^j$$

and substituting this value into (2.7), we have

$$\frac{\overset{\overline{\Lambda}}{\delta}}{\delta t}(\xi_{(\lambda)i} v^i) - \left(\overset{\overline{\Lambda}}{\nabla}_i \xi_{(\lambda)j} + \overset{\overline{\Lambda}}{\nabla}_j \xi_{(\lambda)i}\right) v^i v_j = 0. \tag{2.8}$$

On the basis of (2.5) the equation (2.8) becomes

$$\frac{\overset{\overline{\Lambda}}{\delta}}{\delta t}(\xi_{(\lambda)i} v^i) = \frac{d}{dt}(\xi_{(\lambda)i} v^i) = 0, \tag{2.9}$$

whence it follows directly

$$\xi_{(\lambda)i} v^i = C_{(\lambda)} = \text{const}. \tag{2.10}$$

Thus we have the theorem:

If the quasi-Killing equations (2.5) in a generalized affine geodesic space admit r independent solutions $\xi_{(\lambda)i}$, $(\lambda = 1,2,\dots, r > 0$ then also the differential equations of geodesics (2.6) of the same space admit r independent quasi-linear integrals of the form (2.10).

The integrals (2.10) are not linear in terms of velocities since quasi-Killing vectors already depend on the generalized velocities.

From the generalized statement on first integrals of the differential equations of geodesics it follows that both, Eisenhart's statement on first integrals of differential equations of geodesics of Riemannian spaces and R. Stojanović s statemen, on firts integrals corresponding to the generalized law of moment of momentumı are included as special cases.

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