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UNSTEADY LAMINAR BOUNDARY LAYER ON A ROTATIONAL BODY WHICH IS PUT TO SPIRAL MOTION

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In this paper we shall observe the case of a rotational body which is defined with radius of the transversal crossection $r_0 = r_0\,(x)$ and put to spiral motion. Namely, if we turn upon some rotational body in the plane, which is transversal on the axis, with an angular velocity ω_0, which is changeable in time by law $\omega_0\, t^\beta$, and if we at the same time add into direction of the axis a velocity U_0, which one changes in time by degree law $U_0\, t^\alpha$, then will the resulting motion be spiral, with a walk of the spiral which will be changeable in time.

That means, that we ought to solve the problem of a three-dimensional unsteady laminar boundary layer when

$$\omega\,(t) = \omega_0\, t^\beta ,$$

and if increment of the velocity on the outer edge of the boundary layer along the contour of a rotational body is given by law

$$U(x, t) = U(x)\; U_0\, t^\alpha .$$

For a coordinate system which the x-axis is directed along the generatrix of the contour of a body, and the z-axis on direction of the arc of a transversal crossection, differential equations of the boundary layer for this motion will have the following form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{w^2}{r_0}\frac{dr_0}{dx} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{u\cdot w}{r_0}\frac{\partial r_0}{\partial x} = \nu\frac{\partial^2 w}{\partial y^2}, \tag{2}$$

$$\frac{\partial\,(r_0\, u)}{\partial x} + \frac{\partial\,(r_0\, v)}{\partial y} = 0 . \tag{3}$$

with boundary conditions

$$\begin{aligned} u = v = 0\ , \qquad & w = r_0\,\omega\,(t)\,, \qquad & y = 0\,, \\ u = U(x, t)\,, \qquad & w = 0 \qquad , \qquad & y = \infty . \end{aligned} \tag{4}$$

With $u\,(x, y, t)$, $(v\,x, y\,t)$ and $w\,(x, y, t)$ in the upper equations we have denoted components of the velocity in direction of x, y, respectively z.

If one substitutes the stream function into the upper equations with the expression

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x} - \frac{1}{r_0}\frac{dr_0}{dx}\psi,$$

then the equation 3 will be identically satisfied, and the equations 1 and 2 and the boundary conditions 4 reduced to the new form

$$(1') \qquad \psi_{yt} + \psi_y\psi_{xy} - \psi_x\psi_{yy} - \frac{1}{r_0}\frac{dr_0}{dx}(\psi\psi_{yy} + w^2) = U_t + UU_x + \nu\psi_{yyy},$$

$$(2') \qquad w_t + \psi_y w_x - \psi_x w_y + \frac{1}{r_0}\frac{dr_0}{dx}(\psi_y w - \psi w_y) = \nu w_{yy},$$

$$(4') \qquad \begin{array}{lll} \psi = \psi_y = 0, & w = r_0\,\omega(t), & y = 0, \\ \psi_y = U(x,t), & w = 0, & y = \infty, \end{array}$$

where, with indeces are denoted the partial derivatives by respective coordinates.

If one substitutes the new variable with the expression

$$\eta = \frac{y}{2\sqrt{\nu t}}$$

and if one supposes the form of the stream function and of the function w

$$\psi(x,y,t) = 2\sqrt{\nu t}\,U_0\,U(x)\,t^{\alpha}F(x,\eta,t),$$

$$w(x,y,t) = r_0(x)\,\omega_0\,t^{\beta}\,\Phi(x,\eta,t) \qquad ,$$

then the equations 1'and 2' will be transformed to the new form

$$(1'') \qquad \begin{aligned} &F_{\eta\eta\eta} + 2\eta F_{\eta\eta} + 4\alpha(1 - F_\eta) - 4tF_{\eta t} + 4U_0 t^{\alpha+1}\Big\{U'[1 - F_\eta^2 + FF_{\eta\eta}] + \\ &+ U[F_x F_{\eta\eta} - F_\eta F_{x\eta}] + \frac{r_0'}{r_0}UFF_{\eta\eta}\Big\} + 4\frac{\omega_0^2}{U_0}\frac{r_0 r_0'}{U}t^{2\beta-\alpha+1}\Phi^2 = 0, \end{aligned}$$

$$(2'') \qquad \begin{aligned} &\Phi_{\eta\eta} + 2\eta\Phi_\eta - 4\beta\Phi - 4t\,\Phi_t - 4U_0 t^{\alpha+1}\Big\{U\frac{r_0'}{r_0}[2F_\eta\Phi - F\Phi_\eta] + \\ &+ U[F_\eta\Phi_x - F_x\Phi_\eta] - U'F\Phi_\eta\Big\} = 0, \end{aligned}$$

with boundary conditions

$$(4'') \qquad \begin{array}{lll} F = F_\eta = 0, & \Phi = 1, & \eta = 0, \\ F_\eta = 1, & \Phi = 0, & \eta = \infty. \end{array} \quad \ldots\ldots$$

Now, we can suppose the solution of the upper two-equations in the form of series

$$(5) \qquad F(x,\eta,t) = F_0(\eta) + U_0 t^{\alpha+1}\left[U'F_1(\eta) + U\frac{r_0'}{r_0}F_{1a}(\eta)\right] + $$

$$+ \frac{\omega_0^2}{U_0}\frac{r_0 r_0'}{U}t^{2\beta-\alpha+1}F_{1b}(\eta) + \cdots$$

$$(5') \qquad \Phi(x,\eta,t)=\Phi_0(\eta)+U_0\,t^{\alpha+1}\left[U\frac{r_0'}{r_0}\Phi_1(\eta)+U'\Phi_{1a}(\eta)\right]+\cdots\cdots,$$

and get the system of the usual differential equations for determining unknown functions. These systems have the following forms:

$$\left.\begin{aligned}&F_0'''+2\eta F_0''+4\alpha(1-F_0')=0\;,\\&F_1'''+2\eta F_1''-4(2\alpha+1)F_1'=-4(1-F_0'^2+F_0F_0'')\;,\\&F_{1a}'''+2\eta F_{1a}''-4(2\alpha+1)F_{1a}'=-4F_0F_0''\;,\\&F_{1b}'''+2\eta F_{1b}''-4(2\beta+1)F_{1b}'=-4\Phi_0^2\;,\end{aligned}\right\}\quad\ldots A$$

$$\left.\begin{aligned}&\Phi_0''+2\eta\Phi_0'-4\beta\Phi_0=0\;,\\&\Phi_1''+2\eta\Phi_1'-4(\beta+\alpha+1)\Phi_1=4(2F_0'\Phi_0-F_0\Phi_0')\;,\\&\Phi_{1a}''+2\eta\Phi_{1a}'-4(\beta+\alpha+1)\Phi_{1a}=-4F_0\Phi_0'\;,\end{aligned}\right\}\quad\ldots B$$

with boundary conditions

$$\left.\begin{aligned}&F_0(o)=F_0'(o)=0\;, && F_0'(\infty)=1\\&F_1(o)=F_1'(o)=0\;, && F_1'(\infty)=0\;,\\&F_{1a}(o)=F_{1a}'(o)=0\;, && F_{1a}'(\infty)=0\;,\\&F_{1b}(o)=F_{1b}'(o)=0\;, && F_{1b}'(\infty)=0\;,\end{aligned}\right\}\quad\ldots A'$$

$$\left.\begin{aligned}&\Phi_0(o)=1\;, && \Phi_0(\infty)=0\;,\\&\Phi_1(o)=0\;, && F_1(\infty)=0\;,\\&\Phi_{1a}(o)=0\;, && F_{1a}(\infty)=0\;.\end{aligned}\right\}\quad\ldots B'$$

All equations of both upper systems are the same type, i.e. the linear differential equations of the second order, which can be reduced to Weber's one [2]. The solutions of these differential equations are the functions of the parabolic-cylinder [2], and it can be reduced to Gauss's function of the error, which is introduced with the expression

$$g_\alpha(\eta)=\frac{2}{\sqrt{\pi}\,\Gamma(2\alpha+1)}\int_\eta^\infty(\gamma-\eta)^{2\alpha}e^{-\gamma^2}d\gamma\;.$$

The first two equations of the system A have been solved by Watson [1] for the case when (Ux,t) is given as a degree function in time, and the problem is two-dimensional and non-spiral. The axisymmetric and non-spiral case has been solved by R. Ašković [4] so that the first three equations of the system A are covered with his.

The first differential equation of system A

$$F_0'''+2\eta F_0''-4\alpha F_0'=-4\alpha\;,$$

satisfying boundary conditions, has the following solution

$$F_0'(\eta) = 1 - 2^{2\alpha}\,\Gamma(\alpha+1)\, g_\alpha(\eta) \ ,$$

from where

$$F_0(\eta) = \eta + 2^{2\alpha}\,\Gamma(\alpha+1)\, g_{\alpha+½}(\eta) - \frac{1}{2}\,\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+{}^3/_2)} \ .$$

The second differential equation of the same system

$$F_1''' + 2\eta F_1'' - 4(2\alpha+1)F_1' = -2^{2\alpha+3}\,\Gamma(\alpha+1)(1-\alpha)\,g_\alpha(\eta) + 2^{2\alpha+1}\,\cdot$$
$$\cdot\,\frac{\Gamma^2(\alpha+1)}{\Gamma(\alpha+{}^3/_2)}\, g_{\alpha-½}(\eta) - 2^{2\alpha+1}\,\Gamma(\alpha+1) g_{\alpha-1}(\eta) +$$
$$+ 2^{4\alpha+2}\,\Gamma^2(\alpha+1)\left[g^2_\alpha(\eta) - g^2_{\alpha-½}(\eta)\ \ g_{\alpha+½}(\eta)\right],$$

satisfying the second boundary condition of A', has the following solution

$$F_1'(\eta) = 2^{2\alpha+1}\,\Gamma(\alpha+1)\,\frac{1-\alpha}{1+\alpha}\, g_\alpha(\eta) - 2^{2\alpha-1}\,\frac{\Gamma^2(\alpha+1)}{\Gamma(\alpha+{}^5/_2)}\, g_{\alpha-½}(\eta) +$$
$$+ 2^{2\alpha-1}\,\frac{\Gamma(\alpha+1)}{\alpha+2}\, g_{\alpha-1}(\eta) + 2^{4\alpha+1}\,\Gamma^2(\alpha+1\left[g^2_{\alpha+½}(\eta) - g_\alpha(\eta)\,\cdot\right.$$
$$\left.\cdot\, g_{\alpha+1}(\eta)\right] - 2^{4\alpha+2}\,\Gamma(2\alpha+2)\left[\frac{3-4\alpha}{2+2\alpha} + \frac{2\alpha}{\alpha+2} - \frac{\Gamma^2(\alpha+1)}{\Gamma(\alpha+{}^1/_2)\cdot\Gamma(\alpha+{}^5/_2)} + \right.$$
$$\left. + \frac{1}{2}\,\frac{\Gamma^2(\alpha+1)}{\Gamma^2(\alpha+{}^3/_2)}\right] g_{2\alpha+1}(\eta) \ .$$

The third equation of the system A, which is reduced to the form

$$F'''_{1a} + 2\eta F''_{1a} - 4(2\alpha+1)F'_{1a} = 2^{2\alpha+3}\,\alpha\,\Gamma(\alpha+1)\, g_\alpha(\eta) - 2^{2\alpha+1}\,\Gamma(\alpha+1)\,.$$
$$\cdot\, g_{\alpha-1}(\eta) - 2^{4\alpha+2}\,\Gamma^2(\alpha+1)\, g_{\alpha-½}(\eta)\,\cdot$$
$$\cdot\, g_{\alpha+{}^1/_2}(\eta) + 2^{2\alpha+1}\,\frac{\Gamma^2(\alpha+1)}{\Gamma(\alpha+{}^3/_2)}\, g_{\alpha-{}^1/_2}(\eta),$$

has the following solution

$$F'_{1a}(\eta) = -2^{2\alpha+1}\,\frac{\alpha}{\alpha+1}\,\Gamma(\alpha+1)\, g_\alpha(\eta) - 2^{2\alpha-1}\,\frac{\Gamma^2(\alpha+1)}{\Gamma(\alpha+{}^5/_2)}\, g_{\alpha-½}(\eta) + 2^{2\alpha-1}\,.$$
$$\frac{1}{\alpha+2}\,\Gamma(\alpha+1)\, g_{\alpha-1}(\eta) - 2^{4\alpha+1}\,\Gamma^2(\alpha+1)\, g_\alpha(\eta)\, g_{\alpha+1}(\eta) - 2^{4\alpha+2}\,.$$
$$\Gamma(2\alpha+2)\left[\frac{2\alpha}{\alpha+2} - \frac{4\alpha+1}{2\alpha+2} - \frac{\Gamma^2(\alpha+1)}{\Gamma(\alpha+{}^1/_2)\,\Gamma(\alpha+{}^5/_2)}\right] g_{2\alpha+1}(\eta)$$

As to solve the forth equation of system A we must solve the first equation of the system B. The solution of this equation for the first of the boundary conditions B' is

$$\Phi_0(\eta) = 2^{2\beta}\,\Gamma(\beta+1)\, g_\beta(\eta)\,.$$

Now, we can solve the last equation of the system A.

$$F_{1b}''' + 2\eta F_{1b}'' - 4(2\beta + 1) F_{1b}' = - 2^{4\beta+2}\Gamma^2(\beta+1)\, g^2_\beta(\eta)\,.$$

The solution of this equation which satisfies the last of the boundary conditions A' is

$$F_{1b}'(\eta) = 2^{4\beta+1}\Gamma^2(\beta+1)\frac{\Gamma(2\beta+2)}{\Gamma^2(\beta+{}^3/_2)}\, g_{2\beta+1}(\eta) - 2^{4\beta+1}\Gamma^2(\beta+1)\cdot$$
$$\cdot\; g^2_{\beta+{}^1/_2}(\eta)\,\cdot$$

The second equation of the system B can be reduced to the form

$$\Phi_1'' + 2\eta\Phi_1' - 4(\beta+\alpha+1)\Phi_1 = 2^{2\beta+3}(1+\beta)\Gamma(\beta+1)\, g_\beta(\eta) + 2^{2\beta+1}\Gamma(\beta+1)$$
$$\cdot\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+{}^3/_2)}\, g_{\beta-{}^1/_2}(\eta) - 2^{2\beta+1}\Gamma(\beta+1)\, g_{\beta-1}(\eta) -$$
$$- 2^{2(\beta+\alpha)+3}\Gamma(\alpha+1)\Gamma(\beta+1)\, g_\alpha(\eta)\, g_\beta(\eta) -$$
$$- 2^{2(\beta+\alpha+1)}\,.\,\Gamma(\alpha+1)\Gamma(\beta+1)\, g_{\beta-{}^1/_2}(\eta)\, g_{\alpha+{}^1/_2}(\eta)\,,$$

with the solution

$$\Phi_1(\eta) = 2^{2(\beta+\alpha+1)}\Gamma(\beta+\alpha+2)\left[\frac{5+4\beta}{2+2\alpha} - \frac{\beta}{2\alpha+4} + \frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\beta+{}^1/_2)\Gamma(\alpha+{}^5/_2)} +\right.$$
$$\left.+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+{}^3/_2)\Gamma(\beta+{}^3/_2)}\right] g_{\beta+\alpha+1}(\eta) - 2^{2\beta+1}\frac{1+\beta}{1+\alpha}\Gamma(\beta+1)\, g_\beta(\eta) +$$
$$+ 2^{2\beta-1}\frac{1}{\alpha+2}\Gamma(\beta+1)\, g_{\beta-1}(\eta) - 2^{2\beta-1}\frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+{}^5/_2)}\, g_{\beta-{}^1/_2}(\eta) -$$
$$- 2^{2\beta+2\alpha+1}\Gamma(\alpha+1)\Gamma(\beta+1)\Big[g_\beta(\eta)\, g_{\alpha+1}(\eta) +$$
$$+ 2\, g_{\alpha+{}^1/_2}(\eta)\, g_{\beta+{}^1/_2}(\eta)\Big]\,.$$

The last equation of the system B can be reduced to the form

$$\Phi_{1a}'' + 2\eta\Phi_{1a}' - 4(\beta+\alpha+1)\Phi_{1a} = 2^{2\beta+3}\beta\Gamma(\beta+1)\, g_\beta(\eta) - 2^{2\beta+1}$$
$$\cdot\;\Gamma(\beta+1)\, g_{\beta-1}(\eta) - 2^{2(\beta+\alpha+1)}\Gamma(\beta+1)\Gamma(\alpha+1)\, g_{\beta-{}^1/_2}(\eta)\, g_{\alpha+{}^1/_2}(\eta) +$$
$$+ 2^{2\beta+1}\frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+{}^3/_2)}\, g_{\beta-{}^1/_2}(\eta)\,,$$

with the solution which satisfies the last of the boundary conditions B,

$$\Phi_{1a}(\eta) = 2^{2(\beta+\alpha+1}\Gamma(\beta+\alpha+2)\left[\frac{5+4\beta}{2+2\alpha} - \frac{\beta}{2\alpha+4} + \frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\beta+{}^1/_2)\Gamma(\alpha+{}^5/_2)}\right]\cdot$$
$$g_{\beta+\alpha+1}(\eta) - 2^{2\beta+1}\frac{\beta}{1+\alpha}\Gamma(\beta+1)\, g_\beta(\eta) + 2^{2\beta-1}\frac{1}{\alpha+2}\Gamma(\beta+1.$$
$$g_{\beta-1}(\eta) - 2^{2\beta-1}\frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{\Gamma(\alpha+{}^5/_2)}\, g_{\beta^1-/_2}(\eta) - 2^{2(\beta+\alpha)+1}\,.$$

$$\Gamma(\beta+1)\Gamma(\alpha+1)\, g_\beta(\eta)\, g_{\alpha+1}(\eta)\,.$$

For the components of the skin friction on the surface of a rotational body in direction of x respectively y we have the following expressions

$$\tau_x = \mu \frac{U_0}{2\sqrt{\nu t}} U(x) t^\alpha \Big\{ F_0''(0) + U_0 t^{\alpha+1} \Big[U' F_1''(0) + U \frac{r_0'}{r_0} F_{1a}''(0) \Big] +$$

(6′) $$+ \frac{\omega_0^2}{U_0} \frac{r_0 r_0'}{U} t^{2\beta-\alpha+1} F_{1b}''(0) + \cdots \cdots \Big\},$$

(6′) $$\tau_z = \mu \frac{\omega_0}{2\sqrt{\nu t}} r_0 t^\beta \Big\{ \Phi_0'(0) + U_0 t^{\alpha+1} \Big[U \frac{r_0'}{r_0} \Phi'_1(0) + U' \Phi'_{1a}(0) \Big] + \cdots \Big\}.$$

The position of the curve on the surface of a rotational body along which will arise the separation of the boundary layer from the contour one find from condition $\tau_x = 0$. It follows

(7) $$F_0''(0) + U_0 t^{\alpha+1} \Big[U' F_1''(0) + U \frac{r_0'}{r_0} F_{1a}''(0) \Big] + \frac{\omega_0^2}{U_0} \frac{r_0 r_0'}{U} t^{2\beta-\alpha+1} F_{1b}''(0) + \cdots = 0.$$

The first moment of separation will be denoted with t_s and the way which the body covers during this time

(8) $$s = \int_0^{t_s} U_0 t^\alpha\, dt = \frac{U_0}{\alpha+1} t_s^{\alpha+1}$$

For the case $\alpha = \beta$ it follows that

(7′) $$t_s^{\alpha+1} = - \frac{F_0''(0)}{U_0 \Big[U' F_1''(0) + U \frac{r_0'}{r_0} F_{1a}''(0) \Big] + \frac{\omega_0^2}{U_0} \frac{r_0' r_0}{U} F_{1b}''(0)}$$

and from here

(8′) $$s = \frac{U_0}{\alpha+1} \left\{ - \frac{F_0''(0)}{U_0 \Big[U' F_1''(0) + U \frac{r_0'}{r_0} F_{1a}''(0) + \frac{\omega_0^2}{U_0} \frac{r_0' r_0}{U} F_{1b}''(0) \Big]} \right\}$$

2. In this part we shall observe the case of a rotational body which is put to spiral motion by law

$$U(x,t) = U(x)\, e^{\alpha_1 t}\,,$$

$$\omega(t) = \omega_0\, e^{\beta t}\,.$$

If one substititutes the new variable

$$\eta = y \sqrt{\frac{\alpha}{\nu}}\,,$$

and supposes the forms of functions as

$$\psi(x,y,t) = \sqrt{\frac{\nu}{\alpha}}\, U(x)\, e^{\alpha t}\, F(x,\eta,t) \quad ,$$

$$w(x,y,t) = r_0(x)\, \omega(t)\, \Phi(x,\eta,t) \quad ,$$

then the equations 1 and, are reduced to the shape

$$F\eta\eta\eta - F\eta + 1 - \frac{1}{\alpha} F\eta\, t + \frac{1}{\alpha}\, e^{\alpha t} \Big\{ U'(1 - F_\eta^2 + FF\eta\eta) + U(F_x F\eta\eta - F\eta\, F\eta x) +$$

$$(2,1'') \qquad + U \frac{r_0'}{r_0} FF\eta\eta \Big\} + \frac{\omega_0^2}{\alpha} \frac{r_0 r_0'}{U} e^{(2\beta-\alpha)t} \Phi^2 = 0 \quad ,$$

$$(2.2'') \qquad \Phi\eta\eta - \frac{\beta}{\alpha} \Phi - \frac{1}{\alpha} \Phi_t - \frac{1}{\alpha} e^{\alpha t} \Big\{ \frac{r_0'}{r_0} U (2\,\Phi F\eta - F\Phi\eta) + U(F\eta\, \Phi x -$$

$$- Fx\, \Phi\eta) - U' F \Phi\eta \Big\} = 0 \, .$$

If we suppose the solution in the following form

$$(2,2) \qquad F(x,\eta,t) = F_0(\eta) + \frac{1}{\alpha} e^{\alpha t} \Big[U' F_1(\eta) + U \frac{r_0'}{r_0} F_{1a}(\eta) \Big] +$$

$$\frac{\omega_0^2}{\alpha} \frac{r_0 r_0'}{U} e^{(2\beta-\alpha)t} F_2(\eta) + \cdots$$

$$(2,5') \qquad \Phi(x,\eta,t) = \Phi_0(\eta) + \frac{1}{\alpha} e^{\alpha t} \Big[U \frac{r_0'}{r_0} \Phi_1(\eta) + U' \Phi_{1a}(\eta) \Big] + \cdots ,$$

then the upper equations will be partitioned on a system of the usual differential equations

$$\left.\begin{aligned} & F_0''' - F_0' + 1 = 0, \\ & F_1''' - 2F_1' + (1 - F_0'^2 + F_0 F_0'') = 0, \\ & F_{1a}''' - 2F_{1a}' + F_0 F_0'' = 0, \\ & F_2''' - 2\frac{\beta}{\alpha} F_2' + \Phi_0'^2 = 0, \end{aligned}\right. \qquad A_{\mathrm{I}}$$

$$\left.\begin{aligned} & \Phi_0'' - \frac{\beta}{\alpha}\Phi_0 = 0, \\ & \Phi_1'' - \frac{\alpha+\beta}{\alpha}\Phi_1 - (2\,\Phi_0 F_0' - F_0 \Phi_0') = 0, \\ & \Phi_{1a}'' - \frac{\alpha+\beta}{\alpha}\Phi_{1a} + F_0 \Phi_0' = 0, \end{aligned}\right. \qquad B_{\mathrm{I}}$$

with boundary conditions A' and B'.

The first two equations of the system A_{I} are also in this case covered with those that have been solved by Watson [1] and the third one with that in the paper [4].

The solution of equations of both systems is found very simply so tha it will be only induced here

$$F_0'(\eta) = 1 - e^{-\eta},$$

$$F_1'(\eta) = e^{-\sqrt{2}\,\eta} + (\eta - 1)\, e^{-\eta},$$

$$F_{1a}'(\eta) = \frac{7}{2} e^{-\sqrt{2}\,\eta} + (\eta - 3)\, e^{-\eta} - \frac{1}{2} e^{-2\eta},$$

$$F_2'(\eta) = \frac{1}{2}\left(e^{\sqrt{2\beta/\alpha}\,\eta} - e^{-2\sqrt{\beta/\alpha}\,\eta}\right),$$

$$\Phi_0(\eta) = e^{-\sqrt{\beta/\alpha}\,\eta}$$

$$\Phi_1(\eta) = \left[\frac{3k+2}{2k} + k(2k+1)\right] e^{-\sqrt{1+k}\,\eta} + \left[(2k+1)k - 2 - k\eta\right] e^{-k\eta} + \frac{k-2}{2k} e^{-(1+k)\eta},$$

$$\Phi_{1a}(\eta) = -\left[\frac{1}{2} + k(2k+1)\right] e^{-\sqrt{1+k}\,\eta} + \left[k(2k+1-\eta)\right] e^{-k\eta} + \frac{1}{2} e^{-(1+k)\eta}.$$

For the components of the skin friction in this case we have the following expressions

$$\tau_x = \mu \sqrt{\frac{\alpha}{\nu}}\, U_0\, U_{(x)}\, e^{\alpha t}\left\{ F_0''(0) + \frac{1}{\alpha} e^{\alpha t}\left[U' F_1''(0) + U \frac{r_0'}{r_0} F_{1a}''(0)\right] + \frac{\omega_0^2}{\alpha} \cdot \frac{r_0 r_0'}{U} e^{(2\beta-\alpha)t} F_2''(0) + \dots \right\},$$

$$\tau_z = \mu \sqrt{\frac{\alpha}{\nu}}\, r_0\, \omega_0\, e^{\beta t}\left\{ \Phi_1'(0) + \frac{1}{\alpha} e^{\alpha t}\left[U \frac{r_0'}{r_0} \Phi_1'(0) + U' \Phi_{1a}'(0)\right] + \dots \right\}.$$

For the position of the curve along which will arise the separation of the boundary layer from the contour we will have the following expression

$$F_0''(0) + \frac{1}{\alpha} e^{\alpha t}\left[U' F_1''(0) + U \frac{r_0'}{r_0} F_{1a}''(0)\right] + \frac{\omega_0^2}{\alpha} \frac{r_0 r_0'}{U} e^{(2\beta-\alpha)t} F_2''(0) = 0 .$$

Let us solve now an example: Let the sphere of the radius R put into spiral motion by a sudden jerk. For this case

$$\alpha = \beta = 0; \qquad r_0(x) = R \sin \frac{x}{R} \;;\quad U(x, t) = \frac{3}{2} U_0 \sin \frac{x}{R} \;;\quad \omega(t) = \omega_0 .$$

For the time of separation one obtains the following expression

$$t = -\frac{F_0''(o)}{\left[\frac{3}{2}\frac{U_0}{R}\cos\frac{x}{R}\left(F_1''(o) + F_{1a}''(o)\right) + \frac{\omega_0^2}{U_0}R\cos\frac{x}{R}\cdot\frac{2}{3}F_{1b}''(o)\right]}.$$

The first moment of separation is obtained at $\cos\frac{x}{R} = -1$ i.e. in the last stagnation point

$$t_s = -\frac{F_0''(o)}{\frac{U_0}{R}\left[\frac{3}{2}\left(F_1''(o) + F_{1a}''(o)\right) + \frac{2}{3}\frac{\omega_0^2 R^2}{U_0^2}F_{1a}''(o)\right]}.$$

If one calculates the functions which are to be found in the upper expression for the value of $\eta = 0$ and if one puts the sign

$$\bar{t}_s = \frac{U_0 t_s}{R}, \quad \bar{\omega}_0 = \frac{\omega_0 R}{U_0},$$

then the upper expression for the first moment of separation is reduced to the form

$$\bar{t}_s = \frac{1}{2{,}363055 + 0{,}48407\,\bar{\omega}_0^2}.$$

From where it follows, when the angular velocity is greater, then the separation of the boundary layer from the contour happens more earlier.

LITERATURE

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