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THE STRONG LAW OF LARGE NUMBERS FOR STRICTLY STATIONARY SEQUENCE OF GENERALIZED STOCHASTIC PROCESSES

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1. Introduction. Using generally M. Ullrich's paper [5] we shall prove a strong law of large numbers for sequence of generalized Gelfand's stochastic processes /see [3]/.

The strong law of large numbers for strictly stationary sequence of random variables asserts:

Let $\{\xi_n\}_{n=1}^{\infty}$ be a strictly stationary sequence of random variables for which $E\{|\xi_1|\}<\infty$, and let B be the σ— field of invariant events induced by the sequence $\{\xi_n\}_{n=1}^{\infty}$. Then

$$(a) \qquad \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\xi_j=E\{\xi_1\,|\,B\}$$

with probability 1 (see [1]).

Introducing some new definitions for generalized stochastic processes similar to these for random variables, we shall prove the strong law of large numbers for strictly stationary sequence of generalized stochastic processes.

2. The generalized stochastic process and a sequence of generalized stochastic processes.

Let $K(r)$, $r=1, 2, \cdots$ be a set of all real functions φ of a real variable t which have all derivatives and which are equal to zero out of the interval $|t|\leqslant r$. From [2] follows that $K(r)$ with the usual algebraic operations is a linear separable space, and, therefore, there is a denumerable dense subset $K^{\star}(r)$ with property that for every $\varphi\in K$ there exists a sequence $\{\varphi^{\star}_{i_r}\}_{i_r=1}^{\infty}$ in $K^{\star}(r)$ such that

$$\varphi^{\star}_{i_r}\to\varphi \quad \text{for} \quad i_r\to\infty$$

in the sense of the convergence in $K(r)$.

Let $K=\overset{\infty}{\underset{r=1}{U}}K(r)$ and $K^{\star}=\overset{\infty}{\underset{r=1}{U}}K^{\star}(r)$.

Then K is the space of finite functions φ of the real variable t which have all derivatives and $K^\star = \{\varphi_i^\star\}_{i=1}^{\infty}$ is a denumerable subest of K with property that for every $\varphi_0 \in K$ there exists a sequence

$$\left\{\varphi_{i_0}^\star\right\}_{i_0=1}^{\infty} \qquad \text{in } K^\star \text{ such that}$$

$$(b) \qquad \varphi_{i_0}^\star \longrightarrow \varphi_0 \quad \text{for} \quad i_0 \longrightarrow \infty$$

in the sense of the convergence in K.

Let $/\Omega, \mathrm{A}, P/$ be the probability space and K' the space of generalized functions defined on K. *I. M.* Gelfand defines a generalized stochastic process as a mapping Φ of the probability space $/\Omega, \mathrm{A}, P/$ into the space K' which satisfies the following conditions:

1° for every $\varphi \in K$, $c \in R$ $/R$ is the set of real numbers$/$

$$\{\omega : [\Phi(\omega)](\varphi) < c\} \in \mathrm{A}, \ \omega \in \Omega \ ;$$

2° for every $\omega \in \Omega$, $\varphi_1, \varphi_2 \in K$, $\alpha_1, \alpha_2 \in R$

$$[\Phi(\omega)](\alpha_1 \varphi_1 + \alpha_2 \varphi_2) = \alpha_1 [\Phi(\omega)](\varphi_1) + \alpha_2 [\Phi(\omega)](\varphi_2) \ ;$$

3° for every positive integer m, every set $\left\{\varphi_{i_n}\right\}_{n=1}^{\infty}$, φ_i, $i = 1, 2, \cdots, m$

of functions in K such that $\lim_{n\to\infty} \varphi_{i_n} = \varphi_i$, $i = 1, 2, \cdots, m$

the sequence of m -dimensional distribution functions

$$P(\{\omega : [\Phi(\omega)](\varphi_{i_n}) < c_i, \ i = 1, 2, \cdots, m\})$$

converges for $n \to \infty$ to the distribution function

$$P(\{\omega : [\Phi(\omega)](\varphi_i) < c_i, \ i = 1, 2, \cdots, m\}),$$

for every $(c_1, c_2 \cdots, c_m) \in G$, where G is a dense set in m – dimensonal Euclidean space R^m.

D e f i n i t i o n 1. A generalized stochastic process Φ is measurable with respect to σ -field $\mathrm{B} \subset \mathrm{A}$, or it is B -measurable, if for every $\varphi \in K$

$$\{\omega : [\Phi(\omega)](\varphi) < c\} \in B \ ,$$

where c is a real number.

D e f i n i t i o n 2. The conditional expectation of generalized stochastic process Φ relative to σ -field $\mathrm{B} \subset \mathrm{A}$, $E\{\Phi \mid \mathrm{B}\}$, is a B -measurable stochastic process which staisfies the condition:

$$\int_B E\{[\Phi(\omega)](\varphi) \mid \mathrm{B}\}\, dP = \int_B [\Phi(\omega)](\varphi)\, dP$$

for every $\varphi \in K$, $B \in \mathrm{B}$.

D e f i n i t i o n 3. A sequence of generalized stochastic processes $\{\Phi_n\}_{n=}^{\infty}$ converges to a generalized stochastic process Φ, for $n \longrightarrow \infty$, if for every $\varphi \in K$ and $\omega \in \Omega$ the sequence $\{[\Phi_n(\omega)](\varphi)\}_{n=1}^{\infty}$ converges to $[\Phi(\omega)](\varphi)$ in the sense of convergence in K'.

Definition 4. A sequence of generalized stochastic processes $\{\Phi_n\}_{n=1}^{\infty}$ is equicontinuous at the origin, if for every sequence $\{\varphi_i\}_{i=1}^{\infty}$ in K which converges to 0 for $i \longrightarrow \infty$ and for every $\varepsilon > 0$ there exists an index $i_0(\varepsilon)$ such that for every $i > i_0$ and for every $n = 1, 2, \ldots$

$$(c) \qquad |[\Phi_n(\omega)](\varphi_i)| < \varepsilon .$$

3. The strictly stationary sequence of generalized stochastic processes and the strong law of large numbers.

Definition 1. A sequence of generalized stochastic processes $\{\Phi_n\}_{n=1}^{\infty}$ is strictly stationary if for every $\varphi \in K$, for every set of the indices $i_1, i_2, \cdots, i_m$ (m is an arbitrary positive integer), and for every integer τ

$$(d) \qquad P(\bigcap_{k=1}^{m} \{\omega: [\Phi_{i_k+\tau}(\omega)](\varphi) < c_k\}) = P(\bigcap_{k=1}^{m} \{\omega: [\Phi_{i_k}(\omega)](\varphi) < c_k\}),$$

where $(c_1, c_2, \ldots, c_m)$ is a subset of a dense set G in R^m.

For ω -sets which satisfy (d) we shall say that they are invariant relative to sequence $\{\Phi_n\}_{n=1}^{\infty}$. It can be proved that the class of these sets form a σ -field which we shall name the invariant σ -field relative to sequence $\{\Phi_n\}_{n=1}^{\infty}$.

Theorem 1. /The strong law of large numbers/

Let $\{\Phi_n\}_{n=1}^{\infty}$ be a sequence of generalized stochastic processes which is strictly stationary, equicontinuous at the origin and such that $E\{|\Phi_1|\} < \infty$. Then the sequence

$$(e) \qquad F_n = \frac{1}{n}\sum_{j=1}^{n} \Phi_j, \; n = 1, 2, \ldots$$

converges to $E\{\Phi_1 | \mathrm{B}\}$ for $n \longrightarrow \infty$ with probability 1, where B is the invariant σ -field relative to sequence $\{\Phi_n\}_{n=1}^{\infty}$.

Proof. For every $\varphi_i^* \in K^*$ the sequence $\left\{[\Phi_n(\omega)](\varphi_i^*)\right\}_{n=1}^{\infty}$, is a strictly stationary sequence of random variables for which $E\left\{\left|[\Phi_1(\omega)](\varphi_i^*)\right|\right\} < \infty$ and according to (a)

$$(f) \qquad A_i = \left\{\omega: \lim_{n\to\infty} [F_n(\omega)](\varphi_i^*) = E\left\{[\Phi_1(\omega)](\varphi_i^*) \,\middle|\, \mathrm{B}\right\}\right\} \in \mathrm{A},$$

and $P(A_i) = 1, \quad i = 1, 2, \cdots$

Let us denote $A = \bigcap_{i=1}^{\infty} A_i$. Then $A \in \mathrm{A}$ and $P(A) = 1$.

Let $\omega_0 \in A$ and $\varphi_0 \in K$. Then according to (b) there exists a sequence $\left\{\varphi_{i_0}^*\right\}_{i_0=1}^{\infty}$ in K^* such that $\varphi_{i_0}^* \to \varphi_0$ for $i_0 \to \infty$. Then

$$|[F_n(\omega_0)](\varphi_0) - E\{[\Phi_1(\omega_0)](\varphi_0)|\mathrm{B}\}| \leqslant |[F_n(\omega_0)](\varphi_0) - [F_n(\omega_0)](\varphi_{i_0}^*)| +$$
$$+ |[F_n(\omega_0)](\varphi_{i_0}^*) - E\{[\Phi_1(\omega_0)](\varphi_{i_0}^*)|\mathrm{B}\}| + |E\{[\Phi_1(\omega_0)](\varphi_{i_0}^*)|\mathrm{B}\} - \mathrm{E}\{[\Phi_1(\omega_0)](\varphi_0)|\mathrm{B}\}|.$$

According to the assumption, for a given $\varepsilon > 0$ there is an index i_1 (ε, ω_0) such that for $i_0 > i_1$, $n = 1, 2, \ldots$

$$|[F_n(\omega_0)](\varphi_0) - [F_n(\omega_0)](\varphi^*_{i_0})| \leq \frac{1}{n} \sum_{j=1}^{n} |[\Phi_j(\omega_0)](\varphi_0 - \varphi^*_{i_0})| < \frac{\varepsilon}{3}.$$

According to the convergence of Fatou-Lebesgue (see [4], p. 365), there is an index $i_2(\varepsilon, \omega_0)$ such that for $i_0 > i_2$

$$|E\{[\Phi_1(\omega_0)](\varphi^*_{i_0}) | B\} - E\{[\Phi_1(\omega_0)](\varphi_0) | B\}| < \frac{\varepsilon}{3}.$$

Let us denote $i_3 = \max(i_1, i_2)$ and let $i_0 > 1_3$. According to (f) there is an index $n_0(i_0, \varepsilon)$ such that for $n > n_0$

$$|E\{[\Phi_1(\omega_0)](\varphi^*_{i_0}) | B\} - E\{[\Phi_1(\omega_0)](\varphi_0) | B\}| < \frac{\varepsilon}{3}.$$

Therefore,

$$|[F_n(\omega_0)](\varphi_0) - E\{[\Phi_1(\omega_0)](\varphi_0) | B\}| < \varepsilon \text{ for } n > n_0(i_0, \varepsilon). \tag{g}$$

Since (g) holds for every $\varphi_0 \in K$ it follows that

$\omega_0 \in \bigcap_{\varphi \in K} \{\omega : \lim_{n \to \infty} [F_n(\omega)](\varphi) = E\{[\Phi_1(\omega)](\varphi) | B\}\}$ i. e.

$$A \subset \bigcap_{\varphi \in K} \{\omega : \lim_{n \to \infty} [F_n(\omega)](\varphi) = E[\Phi_1(\omega)](\varphi) | B\}\}.$$

The opposite inclusion is evident, and therefore the set

$$\bigcap_{\varphi \in K} \{\omega : \lim_{n \to \infty} [F_n(\omega)](\varphi) = E\{[\Phi_1(\omega)](\varphi) | B\}\}$$

belongs to the σ-field A and its probability is equal to 1. The theorem is proved.

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