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BUCKLING OF RECTANGULAR PLATES WITH CLAMPED AND SIMPLY-SUPPORTED EDGES

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1. Introduction. For simpy-supported rectangular plates, exact solutions are available for finding the critical buckling load. When all the edges of the plate are clamped, the buckling problem has been solved exactly using multiple Fourier series by Taylor [1] and many other approximate solutions exist/2]. A method is suggested here by which buckling loads for rectangular plates with any combination of fixed and simply-supported edges are determined and by which numerical work is reduced by a considerable amount. The method consists in expressing the deflection function as a double series in characteristic functions describing the normal modes of vibration of beams with similar end conditions. This expression is made to satisfy the differential equation and in so doing we get an infinite determinant which gives the buckling loads. The method has been applied in this paper for the determination of lowest buckling load for a square plate with uniform normal thrust on all the edges, under the following edge conditions: —

(1) All edges clamped
(2) Three sides clamped and the fourth simply-supported
(3) Two adjacent edges clamped and the remaining simply-supported
(4) One edge clamped and the other three simply-supported
(5) Two opposite edges clamped and the other two simply-supported.

The results obtained in some cases are compared with those given by other methods. The convergence of this method is also discussed using the numerical results obtained.

2. The Problem. Consider a rectangular plate of dimensions 2a × 2b shown in Fig. 1. Let P_x and P_y be the uniform axial thrusts in the x and y directions respectively. The differential equation for the elastic transverse displacement of a thin plate under these loads is (Ref. 2, P. 348)

$$(1)\quad D\nabla^4 w + P_x \frac{\partial^2 w}{\partial x2} + P_y \frac{\partial^2 w}{\partial y^2} = 0$$

where

w = transverse deflection of the plate
D = flexural rigidity
∇^4 = biharmonic operator

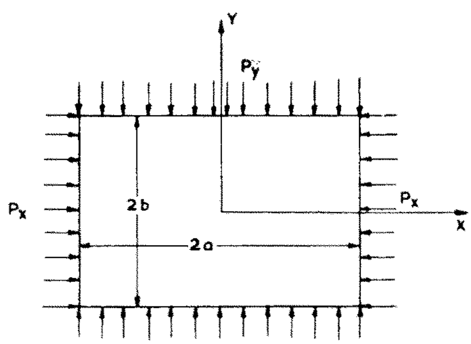

Fig. 1

In all the cases considered here, it is assumed that $Px=Py=P$. Under this type of loading, it is required to determine the lowest critical buckling load P under various edge conditions. If the edge $x=a$ is simply-supported,

(2) $$w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{on } x = a$$

If the same edge is clamped,

(3) $$w = \frac{\delta w}{\partial x} = 0 \quad \text{on } x = a$$

3. The solution. The function for w is chosen in the form

(4) $$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m(x)\, Y_n(y)$$

where $X_m(x)$ and $Y_n(y)$ functions are those which describe the normal modes of vibration of a beam with similar edge conditions.

3.1: — For clamped edge conditions on $x=\pm a$, we have

(5) $$X_m(x) = \frac{\cos h\, \alpha_m x}{\cos h\, \alpha_m a} - \frac{\cos \alpha_m x}{\cos \alpha_m a}$$

where $\alpha_m a$ is determined from the transcendental equation

(6) $$tanh\ \alpha_m a + tan\ \alpha_m a = 0$$

The roots of this equation are: —

(7)

m	$\alpha_m a$
1	2.36504
2	5.49760
3	8.6391
4	11.7806

$$\text{for } m > 4, \quad \alpha_m a \simeq \frac{4m-1}{4}\pi$$

The expression in eqn. (5) is for symmetrical buckling. For antisymmetrical buckling, we have

(8) $$X_m(x) = \frac{\sin h\, \alpha'_m x}{\sin h\, \alpha'_m a} - \frac{\sin \alpha'_m x}{\sin \alpha_m a}$$

where $\alpha'_m a$ is determined from the equation

(9) $$\tanh \alpha'_m a - tan\ \alpha'_m a = 0$$

The roots of this equation are: —

(10)

m	$\alpha'_m a$
1	3.92660
2	7.06858
3	10.21015

$$\text{for } m > 3, \quad \alpha'_m a \simeq \frac{4m+1}{4}\pi$$

3.2: — If the ends $x = \pm a$ are simply-supported, we have

(11) $$X_m(x) = \cos \alpha_m x$$

where

(12) $$\alpha_m a = \frac{2m-1}{2}\pi$$

Similar expressions can be written for $Y_n(y)$. In general, we can write

(13) $$\left.\begin{aligned} X_m(x) &= X_m(\alpha_m x) \\ Y_n(y) &= Y_n(\beta_n y) \end{aligned}\right\}$$

We also know that

(14) $$\left.\begin{aligned} \frac{d^4}{dx^4} X_m(x) &= \alpha_m^4 X_m(x) \\ \frac{d^4}{dy^4} Y_n(y) &= \beta_n^4 Y_n(y) \end{aligned}\right\}$$

To determine the coefficients A_{mn} in equn. (4), this equation is substituted in the differential equation (1) and we get

(15) $$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left[(\alpha_m^4+\beta_n^4) X_m Y_n + 2 X_m'' Y_n'' + \frac{P}{D}\left(X_m'' Y_n + X_m Y_n''\right)\right] = 0$$

where

(16) $$\left.\begin{aligned} X_m'' &= \frac{d^2}{dx^2} X_m(x) \\ Y_n'' &= \frac{d^2}{dy^2} Y_n(y) \end{aligned}\right\}$$

The functions $X_m(x)$ and $Y_n(y)$ are orthogonal in the intervals $(-a, a,)$ and $(-b,b)$ respectively. Hence the functions X_m'' and Y_n'' can be expressed as series in X_m and Y_n respectively. Then

(17) $$\begin{aligned} X_m'' &= \alpha_m^2 \sum_{i=1}^{\infty} K_i^m X_i \\ Y_n'' &= \beta_n^2 \sum_{j=1}^{\infty} L_j^n Y_j \end{aligned}$$

Using this, equation (15) can be written as

(18) $$\sum_m \sum_n A_{mn}\left[(\alpha_m^4+\beta_n^4) X_m Y_n + 2\alpha_m^2 \beta_n^2 \sum_{i=1}^{\infty} K_i^m X_i \sum_{j=1}^{\infty} L_j^n Y_j + \frac{P}{D}\left\{\alpha_m^2 Y_n \sum_{i=1}^{\infty} K_i^m X_i + \beta_n^2 X_m \sum_{j=1}^{\infty} L_j^n Y_j\right\}\right] = 0$$

For the functions given in equations (5) and (8) we have

(19) $$K_i^m = \frac{1}{\alpha_m^2 \cdot 2a}\int_{-a}^{+a} X_m'' X_i \, dx$$

Equation (18) reduces to a set of homogeneous equations for solving A_{mn}. For non-trivial values of A_{mn} the determinant of the coefficient matrix is put to zero. This gives values for the critical loads.

It may be remarked here that whereas in Taylor's method, every term in the series satisfied the differential equation and one set of boundary conditions, the other set being satisfied only approximately if a finite number of terms are taken, in the method proposed in this paper, every term of the double series satisfies the boundary conditions exactly and the differential equation is satisfied approximately by taking only a finite number of terms in the series. The functions chosen here form a complete set in the given interval. It may also be seen that by choosing the expression for w as series of characteristic functions as given here, the application of the Galerkin's method also leads to the same final determinantal equation for getting the buckling loads.

4.1. **Rectangular plate with all edges clamped.**

The boundary conditions are: —

$$\left.\begin{aligned} &\text{On } x = \pm a \qquad w = \frac{\partial w}{\partial x} = 0 \\ &\text{On } y = \pm b \qquad w = \frac{\partial w}{\partial y} = 0 \end{aligned}\right\} \tag{20}$$

Assuming symmetrical buckling, we choose

$$w = \sum_m \sum_n A_{mn} X_m Y_n \tag{21}$$

where

$$\left.\begin{aligned} X_m &= \frac{\cosh \alpha_m x}{\cosh \alpha_m a} - \frac{\cos \alpha^m x}{\cos \alpha_m a} \\ Y_n &= \frac{\cosh \beta_n y}{\cosh \beta_n b} - \frac{\cos \beta_n y}{\cos \beta_n b} \end{aligned}\right\} \tag{22}$$

The values of $\alpha_m a$ and $\beta_n b$ for different values of m and n are given in equn (7). In this case, we have

$$\left.\begin{aligned} K_i^m &= \frac{1}{\alpha_m^2 \, 2a} \int_{-a}^{+a} X_m'' X_i \, dx \\ &= \frac{4 \lambda_i^2}{\lambda_m^4 - \lambda_i^4} \left[\lambda_m \tanh \lambda_m - \lambda_i \tanh \lambda_i \right], \quad i \neq m \\ K_m^m &= \frac{1}{2} \left(\frac{1}{\cosh^2 \lambda_m} - \frac{1}{\cos^2 \lambda_m} \right) + \frac{\tanh \lambda_m}{\lambda_m} \end{aligned}\right\} \tag{23}$$

where $\lambda_m \quad \alpha_m a$

and similar expressions for L_j^n. Because of symmetry in x and y, we have $K_i^m = L_i^m$. The values of K_i^m are given in Table 1.

Table 1. Values of K_i^m

$m \backslash i$	1	2	3	4
1	−0.54984	0.43495	0.34037	0.27302
2	0.08050	−0.81811	0.20139	0.19010
3	0.02551	0.08155	−0.88425	0.12738
4	0.01100	0.04140	0.06850	−0.91512

Substituting equn. (21) in equn. (1), we get the equn. (18). By putting

$$D_{mn} = A_{mn} \left(\alpha_m^4 + \beta_n^4\right)$$

and

$$E_{mn} = \frac{2\,\alpha_m^2\,\beta_n^2}{\alpha_m^4 + \beta_n^4}$$

equation (18) reduces to

$$D_{mn}\left[1 + E_{mn} K_m^m K_n^n + \frac{P}{D}\frac{E_{mn}}{2\beta_n^2} K_m^m + \frac{P}{D}\frac{E_{mn}}{2\alpha_m^2} K_n^n\right]$$

$$+ \sum_{\substack{i \\ j \neq m}} \sum_{j} D_{ij}\, E_{ij}\, K_m^i\, K_n^j + \sum_{\substack{i \\ i \neq m}} D_{in}\left[E_{in}\, K_m^i\, K_n^n + \frac{P}{D}\frac{E_{in}}{2\beta_n^2} K_m^i\right]$$

$$+ \sum_{\substack{j \\ j \neq n}} D_{mj}\left[E_{mj}\, K_m^m\, K_n^j + \frac{P}{D}\frac{E_{mj}}{2\alpha_m^2} K_n^j\right] = 0. \tag{24}$$

Equation (24) gives a set of homogeneous equations for solving D_{mn}. For non-trivial values of D_{mn}, the determinant of the coefficient-matrix should be put to zero. Let us consider a square plate subjected to a uniform edge compression on both sides, i.e., $a = b$ and $P_x = P_y = P$. Hence $E_{mn} = E_{nm}$ and $D_{mn} = D_{nm}$. By taking only one term in each series, the buckling is given by equating the coefficient of D_{11} to zero. It gives

$$K = \frac{Pa^2}{D} = 13.24$$

or

$$P_{cr} = \frac{5.367\,\pi^2 D}{4a^2} \tag{25}$$

Taking $m = 1,2$ and $n = 1$, we get

$$P_{cr} = \frac{5.318\,\pi^2 D}{4a^2} \tag{26}$$

Taking 3 terms, we get

$$P_{cr} = \frac{5.302\,\pi^2 D}{4a^2} \tag{27}$$

An upper bound for the buckling load P_{cr} given by Timoschenko (Ref. 2, P. 387) using energy conditions is $\frac{5.33\pi^2 D}{4a^2}$. The lower bound given by Taylor [1] is $\frac{5.30\pi^2 D}{4a^2}$. Using a collocation method on sixteen points on the boundary, Conway and Leissa [3] obtained a value of $\frac{5.317\pi^2 D}{4a^2}$. Hence it can be seen from the above comparison that using only term in the series in the present method, the buckling load differs from Taylor's value by 1.3%. Taking the second order determinant, the difference is about 0.3%. The third order determinant gives a value which almost coincides with that of Taylor. Hence the series used in this method is very rapidly converging and for all practical purposes, solution of the second order determinant gives sufficiently accurate values for the lowest buckling load. The case when $P_x = 0$ or $P_y = 0$ presents no difficulty. The method is now applied to rectangular plates with other types of edge conditions, to some of which no results exist in the literature.

4.2. **Rectangular plate with one edge simply-supported and the other three clamped** (Fig. 2)

The edge conditions are: —

$$\text{(28)} \quad \begin{cases} \text{On } x = 0 \quad w = \dfrac{\partial^2 w}{\partial x^2} = 0 \\ \text{On } x = 2a \quad w = \dfrac{\partial w}{\partial x} = 0 \\ \text{On } y = \pm b \quad w = \dfrac{\partial w}{\partial y} = 0 \end{cases}$$

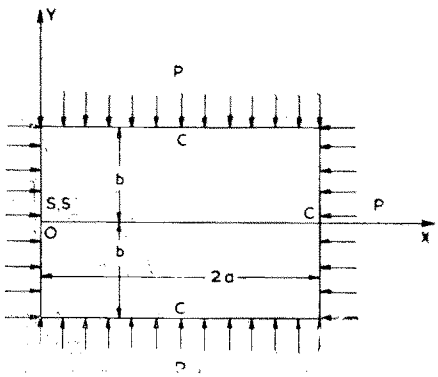

Fig. 2

We choose, in this case

$$X_m(x) = \frac{\sin h\, \alpha_m x}{\sin h\, 2\alpha_m a} - \frac{\sin \alpha_m x}{\sin 2\alpha_m a}$$

$$Y_n(y) = \frac{\cos h\, \beta_n y}{\cos h\, \beta_n b} - \frac{\cos \beta_n y}{\cos \beta_n b}$$

where the values of $2\,\alpha_m a$ and $\beta_n b$ are to be taken from equations (10) and (7) by putting $2\,\alpha_m a$ and $\beta_n b$ for $\alpha_m a$ and $\alpha_m a$ respectively. With these values of $\alpha_m a$ and $\beta_n b$ all the edge conditions in equn. (28) are satisfied. Now proceeding as indicated previously, we have

$$X''_m = \alpha_m^2 \left(\frac{\sin h\, \alpha_m x}{\sin h\, 2\,\alpha_m a} + \frac{\sin \alpha_m x}{\sin 2\alpha_m a} \right)$$

$$Y''_n = \beta_n^2 \left(\frac{\cos h\, \beta_n y}{\cos h\, \beta_n b} + \frac{\cos \beta_n y}{\cos \beta_n b} \right)$$

$$\int_0^{2a} X_m X_i \, dx = \begin{matrix} 0 & if\ m \neq i \\ 2a & if\ m = i \end{matrix}$$

$$\int_{-b}^{+d} Y_n Y_j dy \begin{matrix} = 0 & if\ n \neq j \\ = 2b & if\ n = j \end{matrix}$$

$$K_i^m = \frac{4(2\alpha_i a)^2}{(2\alpha_m a)^4 - (2\alpha_i a)^4}\left[2\alpha_m a \cot h\, 2\alpha_m a - 2\alpha_i a \cot h\, 2\alpha_i a\right] if\ i \neq m.$$

$$K_m^m = \frac{\cot h\, 2\alpha_m a}{2\alpha_m a} - \frac{1}{2}\left(\frac{1}{\sin h^2\, 2\alpha_m a} + \frac{1}{\sin^2 2\alpha_m a}\right)$$

and

$$L_j^n = \frac{4(\beta_j b)^2}{(\beta_n b)^4 - (\beta_j b)^4}\left[\beta_n b \tan h\, \beta_n b - \beta_j b \tan h\, \beta_j b\right] if\ i \neq n$$

(29) $$L_n^n - \frac{1}{2}\left(\frac{1}{\cos h^2 \beta_n b} - \frac{1}{\cos^2 \beta_n b}\right) + \frac{\tan h\, \beta_n b}{\beta_n b}$$

With the substitution $A_{mn}(\alpha_m^4 + \beta_n^4) = D_{mn}$ and $\frac{2\alpha_m^2 \beta_n^2}{\alpha_m^4 + \beta_n^4} = E_{mn}$, Equn. (18) for this case will be

$$D^m{}_n\left[1 + E_{mn} K_m^m L_n^n + \frac{P}{D}\frac{E_{mn}}{2}\left(\frac{K_m^m}{\beta_n^2} + \frac{L_n^n}{\alpha_m^2}\right)\right] + \sum_{\substack{i \\ i \neq m}}\sum_{\substack{j \\ j \neq n}} D_{ij} E_{ij} K_m^i L_n^j$$

(30) $$+ \sum_{\substack{i \\ i \neq m}} D_{in}\left[E_{in} K_m^i L_n^n + \frac{P}{D}\frac{E_{in} K_m^i}{2\beta_n^2}\right] + \sum_{\substack{j \\ j \neq n}} D_{mj}\left[E_{mj} K_m^m L_n^j + \frac{P}{D}\frac{E_{mj} L_n^j}{2\alpha_m^2}\right] = 0.$$

This is an infinite number of simultaneous equations in D_{mn}. The determinant of the coefficient-matrix should be put to zero. Taking a square plate and taking one term we get

(31) $$\frac{4Pa^2}{\pi^2 D} = 4.347$$

Taking two terms in each series such that $m + n \gtrless 3$, thus solving the third order determinant, we have the lowest value

(32) $$\frac{4Pa^2}{\pi^2 D} = 4.322$$

Similarly by solving the sixth order determinantal equation, we have the lowest value of

(33) $$\frac{4Pa^2}{\pi^2 D} = 4.314$$

4.3. **Rectangular plate with two adjacent edges clamped and the remaining two simply-supported** (Fig. 3).

For this case, we choose

$$X_m(x) = \frac{\sin h\, \alpha_m x}{\sin h\, 2\,\alpha_m a} - \frac{\sin \alpha_m x}{\sin 2\,\alpha_m a} \tag{34}$$
$$Y_n(y) = \frac{\sin h\, \beta_n y}{\sin h\, 2\,\beta_n b} - \frac{\sin \beta_n y}{\sin 2\,\beta_n b}$$

with values of $2\,\alpha_m a$ and $2\,\beta_n b$, given by equation (10) by putting $2\,\alpha_m a$ and $2\,\beta_n b$ for $\alpha_m' a$. In this case, K_i^m is the same as given in equn. (29) and further $L_i^m = K_i^m$. Proceeding as in the previous cases, we get, for $a = b$ and for one term in each of the series

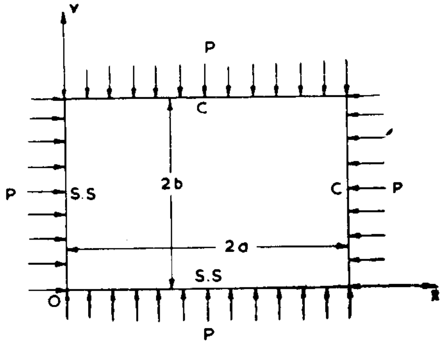

Fig. 3

$$\frac{4\,P\,a^2}{\pi^2\,D} = 3.26 \tag{35}$$

Taking $m = 1{,}2$ and $n = 1$, we get the lowest value of

$$\frac{4\,P\,a^2}{\pi^2\,D} = 3.252 \tag{36}$$

Taking three terms, we get

$$\frac{4\,P\,a^2}{\pi^2\,D} = 3.252 \tag{37}$$

4.4. **Rectangular plate with one edge clamped and the other three simply-supported** (Fig. 4).

In this case, we choose

$$X_m(x) = \frac{\sin h\, \alpha_m y}{\sin h\, 2\,\alpha_m a} - \frac{\sin \alpha_m x}{\sin 2\,\alpha_m a} \tag{38}$$
$$Y_n(y) = \cos \beta_n y$$

with $2\,\alpha_m a$ given by equn. (10) and

$$\beta_n b = \frac{(2\,n-1)\,\pi}{2}$$

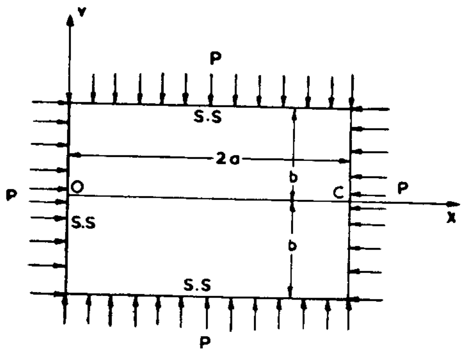

Fig. 4

Proceeding as previous, for a square plate, we get the following results.

	approximation	lowest value of $\frac{4\,P\,a^2}{\pi^2\,D}$
(93)	1st	2.645
	2nd	2.664
	3rd	2.663

4.5. **Rectangular plate with two opposite edges clamped and the other two simply-supported** (Fig. 5).

We take

$$
\left.\begin{aligned}
X_m(x) &= \frac{\cos h\,\alpha_m x}{\cos h\,\alpha_m a} - \frac{\cos \alpha_m x}{\cos \alpha_m a} \\
Y_n(y) &= \cos \beta_n y
\end{aligned}\right\} \qquad (40)
$$

with $\alpha_m a$ given by equation (8) and $\beta_n b = \frac{(2n-1)\pi}{2}$.

For a square plate, taking only one term in each series, we get

$$
\frac{4\,P\,a^2}{\pi^2 D} = 3.837 \qquad (41)
$$

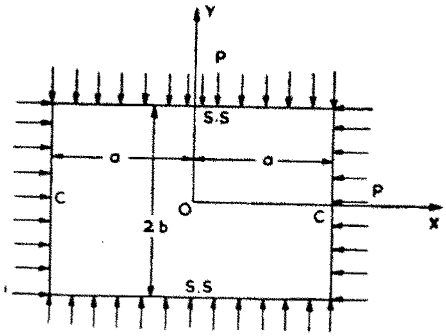

Fig. 5

This case has been worked out by Reissner and as quoted by Taylor, the value of $\frac{4\,P\,a^2}{\pi^2 D}$ is 3.86. Hence it can be seen that in this case only one term gives sufficiently accurate result.

5. Concluding remarks

From the foregoing problems and the numerical results, it can be seen that the method adopted in this paper to calculate the buckling loads is very rapidly convergent. In all the examples treated here, it is assumed that $P_x = P_y = P$. Cases where $P_x = 0$ or $P_y = 0$ or $\frac{P_x}{P_y}$ has any value, do not present any special difficulties. In fact, this proceduere is an extension of Navier's method of treating simply-supported plates to other types of edge conditions. The success of the method depends on the fact that the functions chosen to represent the deflection surface satisfy the edge conditions exactly and also they are orthogonal in the specified interval as such even Galerkin's method of approach using the same functions leads to the same result. Since the function chosen are to satisfy the edge conditions, a free edge cannot be considered exactly in this method, because the beam functions for such conditions do not satisfy the same conditions in a plate. For such problems Rayleigh-Ritz method could be employed using the beam functions. Rectangular plates with elastically restrained edges can also be considered by the method indicated in this paper.

It may be remarked here that a method similar to this has been applied by Lardy (4) for the problem of bending of rectangular plates under transverse load and by Đurić [5] and Hajdin [6] for the two-dimensional elasticity problem. The same method has been successfully applied by the present authors to vibration problems of rectangular plates. An extension of this method to buckling and vibration of orthotropic rectangular plates is straightforward.

REFERENCES

[1] Taylor, G. I. *The buckling load for a rectangular plate with four clamped edges*, Zeitschrift für angewandte Mathematik und Mechanik, Vol. 13, 1933, P. 147.

[2] Timoshenko, S. and Gere, M., *Theory of elastic stability*, McGraw-Hill, 1961.

[3] Conway, H. D. and Leissa, A. W., *A method of investigating certain eigenvalue problems of the buckling and vibration of plates*, Journal of Applied Mechanics, Vol. 27, Series E, No. 3, 1960, P. 557.

[4] Lardy, P., *Sur une méthode nouvelle de résolution du problème des dalles rectangulaires encastrées*, Publications, International Association for Bridge and Structural Engineering, Vol. 13, 1953, P. 197.

[5] Đurić, M., *Solution du problème de la plaque rectangulaire à l'aide des fonctions orthogonales des oscillations transversales d'une poutre*, Publications de l'Institut mathématique, Tome III. 1950. P. 19.

[6] Hajdin, N., *Contribution à la solution du probléme plan*. Publ. de l'Inst. math., Tome V, 1953. p. 53.