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ONE-PARAMETER METHOD FOR CALCULATIONS OF NON-STEADY LAMINAR BOUNDARY LAYERS

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For the present there are not general methods of the Görtler's type, of the type of exact solutions, for solving the problem of non-steady laminar boundary layers, but there are approximate methods. Namely, Struminsky and Rozin in their studies [1], [2] have shown the possibility of applying ideas of one-parameter methods of steady boundary layers on non-steady ones, where Rozin has come to the practically applicable solution. But, although the problem has been linearized, i.e.has been reduced to linear partial equation, there still remained the arithmetical difficulty. Beside this method, it is necessary to mention also Targ's method, [2] which is not bound on momentum equations.

In this paper, as it will be further shown, it has been attempted, starting from momentum equations, to arrive at a simpler solution of single approximations, represented by ordinary integrals .The procedure of practical solution of the problem is more simpler and shorter, inasmuch as a little less exact, because there is the possibility of solving only equations for the first two approximations.

The differential equations and momentum equations

The differential equations of unsteady plain laminary boundary layers have the following form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2}$$

with boundary and supplementary boundary conditions

$$\begin{aligned} &u=v=0;\ \frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}=-\nu\frac{\partial^2 u}{\partial y^2};\quad y=0,\\ &u=U;\quad \frac{\partial u}{\partial y}=0\ ;\quad \frac{\partial^2 u}{\partial y^2}=0;\qquad y\to\infty \end{aligned} \tag{3}$$

where

x — the distance along the wall of the contour which is encircled by the fluid;

y -- the vertical distance from the wall of the contour;

t — time;

$u\,(x, y, t)$ and $v\,(x, y, t)$ — the component of the velocity of the current in x respectively y in the direction;

ν — the kinetic viscosity;

$U\,(x, t)$ — the velocity of the external potential current outside the boundary layer.

On the base of the physical justification, representing the velocity of the current within the boundary layer in the expression

$$u\,(x, y, t) = u_0\,(x, y, t) + u_1\,(x, y, t) + \ldots$$

we shall obtain the differential equations for the first and for the second approximation. The differential equations for the first approximation have the following form

$$(1^0) \qquad \frac{\partial u_0}{\partial t} = \frac{\partial U}{\partial t} + \nu \frac{\partial^2 u_0}{\partial y^2},$$

$$(2^0) \qquad \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0,$$

with boundary and supplementary conditions

$$(3^0) \qquad \begin{aligned} & u_0 = 0; \quad \frac{\partial U}{\partial t} = -\nu \frac{\partial^2 u_0}{\partial y^2}, && \text{for y} = 0, \\ & u_0 = U; \quad \frac{\partial u_0}{\partial y} = 0; \quad \frac{\partial^2 u_0}{\partial y^2} = 0, && \text{for y} = \infty, \end{aligned}$$

while for the second approximation we have the following differential equations

$$(1^1) \qquad \frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial y^2} = U \frac{\partial U}{\partial x} - u_0 \frac{\partial u_0}{\partial x} - v_0 \frac{\partial u_0}{\partial y},$$

$$(2^1) \qquad \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0,$$

with boundary and supplementary conditions

$$(3^1) \qquad \begin{aligned} & u_1 = 0; \quad U \frac{\partial U}{\partial x} = -\nu \frac{\partial^2 u_1}{\partial y^2}, && \text{for y} = 0, \\ & u_1 = 0; \quad \frac{\partial u_1}{\partial y} = 0; \quad \frac{\partial^2 u_1}{\partial y^2} = 0, && \text{for y} = \infty. \end{aligned}$$

Integrating the differential equations by the thickness of the boundary layer, and introducing the displacement thickness $\delta^\star$, the momentum thickness $\delta^{\star\star}$ and the skin friction τ with expressions

$$(4) \qquad \delta^\star = \int_0^\infty \left(1 - \frac{u}{U}\right) dy = \int_0^\infty \left(1 - \frac{u_0}{U}\right) dy - \int_0^\infty \frac{u_1}{U}\, dy = \delta_0^\star + \delta_1^\star,$$

$$\delta^{**}=\int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy=\int_0^\infty \frac{u_0}{U}\left(1-\frac{u_0}{U}\right)dy-2\int_0^\infty \frac{u_0}{U}\frac{u_1}{U}\,dy-\int_0^\infty \frac{u_1}{U}\left(1-\frac{u_1}{U}\right)dy=\delta_0^{**}+\delta_1^{**},\tag{5}$$

$$\tau=\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}=\mu\left(\frac{\partial u_0}{\partial y}\right)_{y=0}+\mu\left(\frac{\partial u_1}{\partial y}\right)_{y=0}=\tau_0+\tau_1,\tag{6}$$

we have the momentum equations for the two cited approximations

$$\frac{\partial}{\partial t}(U\,\delta_0^{*})=\frac{1}{\rho}\tau_0,\tag{7}$$

$$\frac{\partial}{\partial t}(U\,\delta_1^{*})+U^2\frac{\partial\,\delta_0^{**}}{\partial x}+U\frac{\partial U}{\partial x}(\delta_0^{*}+2\,\delta_0^{**})=\frac{1}{\rho}\tau_1.\tag{8}$$

The transformation of momentum equations

For the transformation of the first of the two momentum equations, we shall suppose that the velocity u_0 is the function of the parameter f_0.

$$\frac{u_0(x,y,t)}{U(x,t)}=\varphi(\eta;f_0),\tag{9}$$

where
$$\eta=\frac{y}{\delta_0^{*}}\,.$$

We shall determine the parameter f_0 from the condition

$$f_0=\left(\frac{d^2\varphi}{d\eta^2}\right)_{\eta=0}=\frac{\delta_0^{*2}\,U_t}{\nu\,U},\tag{10}$$

where, and this will be also in future, with indeces denoted, the partial derivative by respective coordinates.

Now, the relation of the momentum thickness and of the displacement thickness is

$$\frac{\delta_0^{**}}{\delta_0^{*}}=\int_0^\infty \varphi(1-\varphi)\,d\eta=H_0^{**}(f_0),\tag{11}$$

and for the skin friction is obtained the expression

$$\tau_0=\mu\left(\frac{\partial u_0}{\partial y}\right)_{y=0}=\frac{\mu\,U}{\delta_0^{*}}\zeta_0(f_0).\tag{12}$$

Because of expressions 10, 12, the equation 7 is transformed into the form

$$f_{0t}+f_0\left(3\frac{U_t}{U}-\frac{U_{tt}}{U_t}\right)=2\frac{U_t}{U}\zeta_0.\tag{13}$$

For the transformation of the equation 8, we shall suppose that the velocity u_1 is the function of the parameter f_1.

$$(14) \qquad \frac{u_1(x,y,t)}{U(x,t)} = \varphi_1(\eta_1; f_1),$$

where $\eta_1 = \frac{y}{\delta_1^{\star}}$, and f_1 — is the new parameter which we shall determine from the condition

$$(15) \qquad f_1 = -\left(\frac{d^2\varphi_1}{d\eta_1^2}\right)_{\eta_1=0} = \frac{\delta_1^{\star 2} U_x}{\nu}.$$

Now it is

$$(16) \qquad \tau_1 = \mu\left(\frac{\partial u_1}{\partial y}\right)_{y=0} = \frac{\mu U}{\delta_1^{\star}} \zeta_1(f_1).$$

Because of expressions 10, 11, 15 and 16, the equations 8 is transformed into the form

$$(17) \qquad f_{1t} + f_1\left(2\frac{U_t}{U} - \frac{U_{xt}}{U_x}\right) + \sqrt{\frac{f_1}{f_0}}\sqrt{\frac{UU_x}{U_t}}\, U_x \left\{\left[f_{0x}\frac{U}{U_x} + f_0\left(1 - \frac{UU_{xt}}{U_x U_t}\right)\right] H_0^{\star\star} + 2\frac{U}{U_x}\frac{dH_0^{\star\star}}{df_0} f_0 f_{0x} + 2 f_0 (1 + 2H_0^{\star\star})\right\} = 2 U_x \zeta_1.$$

By this way, we have obtained two differential equations for the determination of parameters f_0 and f_1 and of unknown functions $\zeta_0(f_0)$ and $\zeta_1(f_1)$.

The determination of unknown functions

To determine the functions $\zeta_0(f_0)$ and $\zeta_1(f_1)$ it ought to be given some determined profile of velocity on the intersection of the boundary layer. We shall take the Pohlhausen's type of profile

$$(19') \qquad \frac{u_0}{U} = \varphi(\eta; \lambda_0) = a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4,$$

where $\eta = \frac{y}{\delta}$, and λ_0 -local parameter of the form which is determined from the condition

$$(10') \qquad \lambda_0 = -\left(\frac{d^2\varphi}{d\eta^2}\right)_{\eta=0} = \frac{\delta^2 U_t}{\nu U}.$$

Satisfying the boundary and supplementary conditions, we shall obtain the profile of velocities in the form

$$('') \qquad \frac{u_0}{U} = 1 - (1+\eta)(1-\eta)^2 + \frac{\lambda_0}{6}\eta(1-\eta)^3.$$

Now, on the base of 9″, 4 and 5, the displacement thickness and the momentum thickness are for the first approximation

$$\delta_0^\star = \int_0^\delta \left(1 - \frac{u_0}{U}\right) dy = \delta \left(\frac{3}{10} - \frac{\lambda_0}{120}\right),$$

$$\delta_0^{\star\star} = \int_0^\delta \frac{u_0}{U}\left(1 - \frac{u_0}{U}\right) dy = \delta \left(\frac{37}{315} - \frac{\lambda_0}{945} - \frac{\lambda_0^2}{9072}\right),$$

or

(18) $$\frac{\delta_0^\star}{\delta} = \frac{3}{10} - \frac{\lambda_0}{120},$$

(19) $$\frac{\delta_0^{\star\star}}{\delta} = \frac{37}{315} - \frac{\lambda_0}{945} - \frac{\lambda_0^2}{9072}.$$

We find the skin friction from the expression

$$\tau_0 = \mu \left(\frac{\partial u_0}{\partial y}\right)_{y=0} = \mu \frac{U}{\delta}\left(2 + \frac{\lambda_0}{6}\right),$$

and from there

(20) $$\frac{\tau_0 \delta}{\mu U} = 2 + \frac{\lambda_0}{6} = b(\lambda_0).$$

From expressions 10 and 10′ we can find easily the connexion of the parameter f_0 and λ_0.

(21) $$f_0 = \left(\frac{3}{10} - \frac{\lambda_0}{120}\right)^2 \lambda_0.$$

From 20 on account of 18 is obtained the expression for the characteristic function $\zeta_0 = \zeta_0(\lambda_0)$.

(22) $$\zeta_0 = \left(2 + \frac{\lambda_0}{6}\right)\left(\frac{3}{10} - \frac{\lambda_0}{120}\right) = \zeta_0(\lambda_0),$$

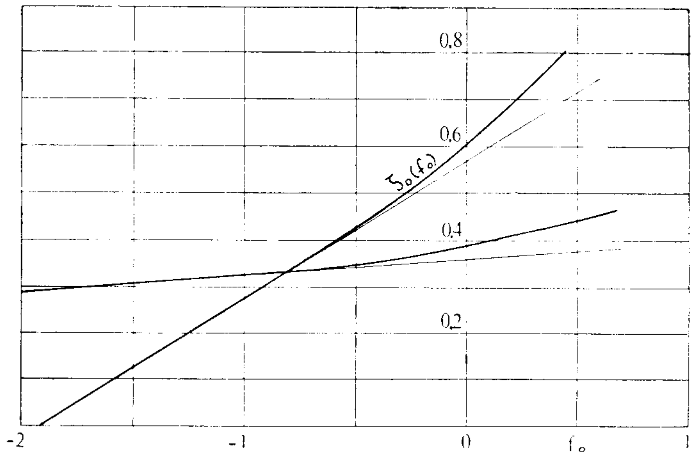
and because of the connexion 21

$$\zeta = \zeta_0(f_0).$$

For further work we need also the function $H_0^{\star\star}(f_0)$ which we shall determine after the expression 11 as the relation of the momentum thickness and the displacement thickness

(23) $$H_0^{\star\star} = \frac{\int_0^1 \varphi(1-\varphi)\, d\eta}{\int_0^1 (1-\varphi)\, d\eta} = \frac{\frac{37}{315} - \frac{\lambda_0}{945} - \frac{\lambda_0^2}{9072}}{\frac{3}{10} - \frac{\lambda_0}{120}}.$$

The functional dependence $\zeta_0 = \zeta_0\,(f_0)$ and $H_0^{**} = H_0^{**}\,(f_0)$ in the form of the diagram



We shall also assume analogously the profile of velocity for the second approximation in the form of the polynomial of the fourth degree

$$\frac{u_1}{U} = \varphi_1\,(\eta;\,\lambda_1) = b_1\,\eta + b_2\,\eta^2 + b_3\,\eta^3 + b_4\,\eta^4, \tag{14'}$$

where $\eta = \frac{y}{\delta}$, and λ_1 – is the convective parameter of the form which is determined by conditions

$$\lambda_1 = -\left(\frac{d^2\,\varphi_1}{d\eta^2}\right)_{\eta=0} = \frac{\delta^2\,U_x}{\nu}. \tag{15'}$$

Satisfying boundary and supplementary conditions, we obtain the family profiles for the second approximation

$$\frac{u_1}{U} = \frac{\lambda_1}{6}\,\eta\,(1-\eta)^3. \tag{14''}$$

Now, on the base of the expressions 14″, 4 and 5 we can give the displacement thickness, the momentum thickness and the skin friction in the function of the given parameter

$$\delta_1^{*} = -\,\delta\frac{\lambda_1}{120},$$

$$\delta_1^{**} = \delta\left(-\frac{\lambda_1}{945} - 2\,\frac{\lambda_0\,\lambda_1}{9072} - \frac{\lambda_1^2}{9072}\right),$$

or

$$\frac{\delta_1^{*}}{\delta} = -\,\frac{\lambda_1}{120}, \tag{24}$$

(25) $$\frac{\delta_1^{**}}{\delta} = -\frac{\lambda_1}{945} - 2\frac{\lambda_0 \lambda_1}{9072} - \frac{\lambda_1^2}{9072},$$

and the skin friction

$$\tau_1 = \mu\left(\frac{\partial u_1}{\partial y}\right)_{y=0} = \frac{\mu U}{\delta}\frac{\lambda_1}{6},$$

respectively

(26) $$\frac{\tau_1 \delta}{\mu U} = \frac{\lambda_1}{6}.$$

From the expressions 15 and 15′ we find the connexion of the parameter f_1 and λ_1

(27) $$f_1 = \frac{\lambda_1^3}{120^2}$$

The characteristic function ζ_1 we find from the derivatives 26 and 24

(28) $$\zeta_1 = -\frac{\lambda_1^2}{6 \cdot 120} = \zeta_1(\lambda_1),$$

or, because of 27

$$\zeta_1 = \zeta_1(f_1).$$

The linearization of equations

As one sees from the diagram, the characteristic function $\zeta_0(f_0)$ deviates ittle from the straight line, so, it can be approximated with that one

(29) $$\zeta_0(f_0) = a + b f_0.$$

Because of the cited approximation, the equation 13 receives the form

(13′) $$f_{0t} + f_0\left[(3-2b)\frac{U_t}{U} - \frac{U_{tt}}{U_t}\right] = 2a\frac{U_t}{U}$$

with the solution

$$f_0 = \frac{2a\,U_t}{U^{3-2b}}\int_0^t U^{3-2b-1}\,dt + C\frac{U_t}{U^{3-2b}}.$$

If it is, as it exists at the flow past a body. U = 0 at, t = 0, then, for the aim of the finality at t = 0, it must be C = 0, and the solution receives the final form

(30) $$f_0 = \frac{2a\,U_t}{U^{3-2b}}\int_0^t U^{3-2b-1}\,dt \quad .$$

To solve the equation 17, we introduce instead of the parameter f_1, the parameter λ_1 through the connexion 27 and finding the relation of the parameter λ_0 and λ_1 through 10′ and 15′ we shall reduce the equation 17 to its new form

$$(17') \qquad \lambda_{1t} - \lambda_1 \left(\frac{2}{3} \frac{U_t}{U} - \frac{1}{3} \frac{U_{xt}}{U_x} \right) - 40\, U_x \left[\frac{1}{3} + F(f_0) \right] = 0,$$

where

$$(31) \qquad F(f_0) = \frac{1}{\sqrt{f_0 \lambda_0}} \left\{ \left[\frac{f_{ox}}{f_o} \frac{U}{U_x} + \left(1 - \frac{U\, U_{xt}}{U_x\, U_t} \right) \right] f_0\, H_0^{\star\star} + \right.$$
$$\left. + 2 \frac{U}{U_x} \frac{d H_0^{\star\star}}{d f_0} f_0\, f_{0x} + 2 f_0 (1 + 2\, H_0^{\star\star}) \right\}$$

As one sees from the diagram, the function $H_0^{\star\star}$ depends on slightly of f_0, and so one can put

$$\frac{d H_0^{\star\star}}{d f_0} \approx 0,$$

and because of this, the expression 31 is reduced to the form

$$(31') \qquad F(f_0) = \sqrt{\frac{f_0}{\lambda_0}} \left\{ \left[\frac{f_{0x}}{f_0} \frac{U}{U_x} + \left(1 - \frac{U U_{xt}}{U_x U_t} \right) \right] H_0^{\star\star} + 2 (1 + 2\, H_0^{\star\star}) \right\}.$$

The solution of the equation 17′ is

$$\lambda_1 = 40 \frac{U_x^{1/3}}{U^{2/3}} \int_0^t (U U_x)^{2/3} \left[\frac{1}{3} + F(f_0) \right] dt + C_1 \frac{U_x^{1/3}}{U^{2/3}} \quad .$$

If at $t=0$, the velocity is $U=0$, then, for the finality of the solution at $t=0$, it must be $C_1=0$, and therefore

$$(32) \qquad \lambda_1 = 40 \frac{U_x^{1/3}}{U^{2/3}} \int_0^t (U U_x)^{2/3} \left[\frac{1}{3} + F(f_0) \right] dt \quad ,$$

or, if we return to the parameter f_1 through the connexion 27

$$(33) \qquad f_1 = \frac{40}{9} \frac{U_x}{U^2} \left\{ \int_0^t (U U_x)^{2/3} \left[\frac{1}{3} + F(f_0) \right] dt \right\}^3 .$$

If the function of velocity of the external potential current is given in the form which will appear most frequently,

i.e.

$$U(x,\, t) = U(x) \cdot \Omega(t) \;,$$

then that simplifies much the solutions

$$(30') \qquad f_0 = \frac{2\, a\, \dot{\Omega}}{\Omega^{3-2b}} \int_0^t \Omega^{3-2b-1}\, dt \quad ,$$

$$(33') \qquad f_1 = \frac{40}{9} \frac{U'^3}{U} \left\{ \int_0^t \Omega^{4/3} \left[\frac{1}{3} + F(f_0) \right] dt \right\}^3 \quad ,$$

where $U = U_{(x)}$ – the function of external velocity at the stationary state. $\Omega = \Omega(t)$ – function which shows the character of unsteadiness. With $(\dot{})$ respectively $(')$ is designed the differentiation by t respectively by x.

The function $F(f_0)$ for this case is reduced to the form

$$F(f_0) = 2\sqrt{\frac{f_0}{\lambda^0}}(1 + 2H_0^{**}) = 2\cdot(0{,}53492 - 0{,}0104497\ \lambda_0 - 0{,}0002204\ \lambda_0^2),$$

because $f_{0x} = 0$. Respectively, because of the connexion 21 one obtains the final form for this function..

$$F(f_0) = 2\ (0{,}53492 + 0{,}02646 f_0 - 0{,}738\ f_0^2). \tag{31''}$$

We find the point of separation from the expression for the skin friction

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0,$$

of what is

$$f_0 + f_1 = -1{,}92\,. \tag{34}$$

We shall now determine the regularity of constants a and b. One sees from the diagram that the curve $\zeta(f_0)$ deviates little from the straight line. We shall ask that this deviation ought to be the least in the most important field of change of the parameter f_0 i.e. in the interval $f_0 = -1.92$ to $f_0 = 0$. And this is just the interval that the point of separation passes from the first moment of separation to the attainment of the steady state. The minimal deviation of the curve from the straight line in this interval gives the values of constants

$a = 0{,}58\,,\qquad b = 0{,}3\,.$

Now, finally we can write the solution which is practically useful

$$f_0 = \frac{1{,}16\ \Omega}{\Omega^{2,4}}\int_0^t \Omega^{1,4}\,dt, \tag{30''}$$

$$f_1 = \frac{40}{243}\frac{U'^3}{\Omega}\left\{\int_0^t \Omega^{4/3}\,[1 + 6\,(0{,}53492 + 0{,}02646 f_0 - 0{,}738\,f_0^2]\,dt\right\}^3, \tag{33''}$$

$$f_0 + f_1 = -1{,}92\,. \tag{34'}$$

From the equation 34′ for every moment of time we can determine the position of the point of separation. Generally, in practical problems the interesting question is the first moment of separation of the boundary layer from the contour.

Application

1. Let us observe the case of the cylinder which is put rapidly into motion For this case is

$$U(x,t) = U(x) = 2\,U_\infty\, sin\frac{x}{U}\,.$$

From the condition 34′ one obtains for the first moment of partition

$$\text{tod} = 0{,}275\,\frac{R}{U_\infty}\,,$$

respectively the trajectory performed by the cylinder till that moment

$Sod = tod \cdot Uoo = 0{,}275 \cdot R.$

By the Struminsky-Rozin method [2] the obtained value is

$Sod = 0{,}300 \ R.$

By the Targ method [2]

$Sod = 0{,}250 \ R.$

By Goldstein and Rosenhed [3] the obtained value is

$Sod = 0{,}320 \cdot R.$

2. Let bring now the cylinder into uniform motion, then

$$U(x, t) = U(x) \cdot t = 2b \cdot sin \frac{x}{R} \cdot t,$$

and from the condition 34′ one obtains

$$tod = 0{,}94 \frac{R}{b},$$

and for the trajectory performed by the cylinder till the moment of separation. the obtained value is

$Sod = 0{,}44 \ R.$

Blazius by a mathematically stronger method has obtained the value

$$tod = 1{,}02 \frac{R}{b},$$

$$Sod = 0{,}52 \ R.$$

The comparison of methods and some conclusions

In this part we shall make some observations which have been observed on these some examples. Namely, the question is about the following. For the case of the cylinder which is put into motion by a sudden jerk, Blasius, stopping at the second approximation, obtained the value for the first moment of time of separation

$$tod = 0{,}351 \frac{R}{U_\infty},$$

and Godstein and Rozenhed, stopping at the third approximation, obtained the value

$$tod = 0{,}32 \frac{R}{U_\infty},$$

that makes the difference of about 9%.

By the Struminsky-Rozin method, the obtained value is

$$tod = 0{,}300 \frac{R}{U_\infty},$$

and we have obtained

$$tod = 0{,}275 \frac{R}{U_\infty},$$

which represents a deviation of about 9%.

If the Struminsky-Rozin method is taken as the exact one from these two approximate methods, then one sees that the deviation of our method is about 9%, and also this deviation would have been less if for example the Loitsianskii-Kotchin or the Falkner-Skan profile had been taken, as in the case of the Struminsky Rozin method. In the class of exact solutions the deviation between the second and the third approximation is also 9%. These results should be verified on a much greater number of examples. But, one can still make the assertion, though not to be too quick in making generalization before it would have been demonstrated on a much greater number of examples, that it would be already sufficient to stop at the third approximation, so that the error would not be too great. This means, and this is probably also right, that already the third approximation would bring us quite in the domain of exact values. Thus, in the case of exact methods, it would be probably sufficient to stop at the third approximation.

If spite of the fact that this method gives about 9% less good results than the Struminsky-Rozin method, for practical calculations, where one should come quickly to results, this method could applied quite successfully.

The case of rotational bodies

Let us observe now the unsteady laminar boundary layers on rotational bodies which are defined by the radius of the transversal crossection $r = r(x)$. For this case the differential equations have the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{2.1}$$

$$\frac{\partial (r u)}{\partial x} + \frac{\partial (r v)}{\partial y} = 0, \tag{2.2}$$

with boundary and supplementary boundary conditions

$$\begin{aligned} &u = v = 0\,; \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\nu\frac{\partial^2 u}{\partial y^2}\,; \text{ for } y = 0\,, \\ &v = U\,; \quad \frac{\partial u}{\partial y} = 0\,; \quad \frac{\partial^2 u}{\partial y^2} = 0\,,\ y = \infty\,. \end{aligned} \tag{2.3}$$

Analogously, as in the case of plane boundary layers, we shall start also here from the differential equations for the first two approximations, and write the momentum equations for these two approximations

$$\frac{\partial (U\delta_0{}^\star)}{\partial t} = \frac{1}{\rho}\tau_0\,, \tag{2.7}$$

$$\frac{\partial (U\delta_1{}^\star)}{\partial t} + U^2\frac{\partial \delta_0{}^{\star\star}}{\partial x} + U\frac{\partial U}{\partial x}(\delta_0{}^\star + 2\,\delta_0{}^{\star\star}) + \frac{r'}{r}U^2\delta_0{}^{\star\star} = \frac{1}{\rho}\tau_1. \tag{2.8}$$

If we compare these two equations with those plane cases, we shall see that the only difference is in the momentum equation for the second approximation., That means, that for the first approximation the solutions will be the same as at the plane boundary layer, while for the second, the form of the solution will be the same, and the difference will appear in the form of the function $F(f_0)$. Namely, in this case, beside the function $F(f_0)$ will appear also a supplementary member, i. e., it will be

$$F_1(f_0) = F(f_0) + \frac{r'}{r}\frac{U}{U_x}\,2\sqrt{\frac{f_0}{\lambda_0}}\,H_0{}^{\star\star},$$

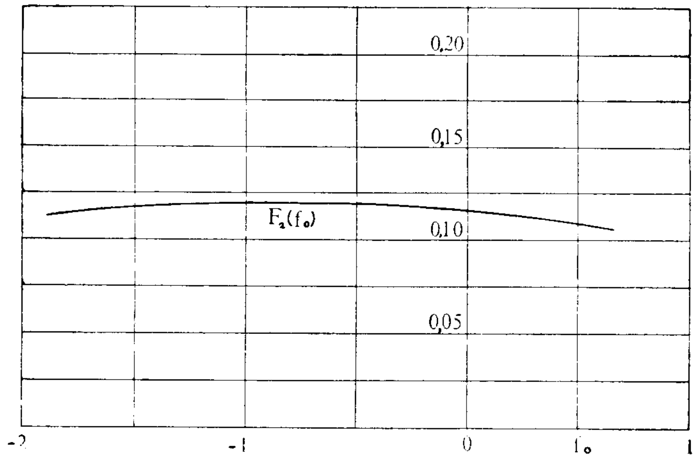
or

$$F_1(f_0) = F(f_0) + 2\,\frac{r'}{r}\frac{U}{U_x}F_2(f_0)\,.$$

For obtaining final solutions, it is necessary to investigate the function $F(f_0)$. We shall present it graphically. Because of the connexions 21 and 23, the function F_2 will have the form

$$F_2(\lambda_0) = \frac{37}{315} - \frac{\lambda_0}{945} - \frac{\lambda_0^2}{9072},$$

while the graphic the following



One sees from the diagram that the function F_2 depends slightly on f_0 and so, approximately, we can take it as the constant

$$F_2 = \alpha .$$

Now, we can finally write the solution, and this at once for the case when $U(x, t) = U(x) \cdot \Omega(t)$. It follows

$$\text{(2.30')} \qquad f_0 = \frac{1{,}16\,\dot{\Omega}}{\Omega^{2,4}} \int_0^t \Omega^{1,4}\, dt ,$$

$$f_1 = \frac{40}{243} \frac{U'^3}{\Omega} \left\{ \int_0^t \Omega^{4/3} \left[1 + 6\,(0{,}53492 + 0{,}0264\, f_0 - \right.\right.$$

$$\text{(2.33')} \qquad \left.\left. - 0{,}738\, f_0^2) + 6\,\alpha \frac{r'}{r} \frac{U}{U'} \right] dt \right\}^3 ,$$

and we find the separation point also from the the condition

$$\text{(2.34)} \qquad f_0 + f_1 = -1{,}92 .$$

For the value of the constant α one can take the approximative mean value

$$\alpha = 0{,}1180778 .$$

Let us solve now an example: Let the sphere of the radius R put into motion by a sudden jerk. For this case

$$r(x) = R \sin \frac{x}{R}, \quad U(x, t) = \frac{3}{2} U_\infty \ \sin \frac{x}{R},$$

From the condition 2.34 taking into account the expressions 2.30′ and 2.33′ one obtains for first moment of time of separation the value

$$\text{tod} = 0{,}30757 \ \frac{R}{U_\infty} \ .$$

TABLES OF CHARACTERISTIC FUNCTIONS

λ_0	f_0	ζ_0	H_0^{**}	$\frac{1}{2} F$	F_0
13	0,4772	0,7985			
12	0,4800	0,8000	0,44500	0,37784	
11	0,4774	0,7989	0,44434	0,39335	
10	0,4694	0,7944	0,44255	0,40837	
9	0,4556	0,7875	0,44015	0,42305	
8	0,4355	0,7777	0,43776	0,43724	
7	0,4086	0,7653	0,43290	0,45097	
6	0,3750	0,7500	0,42892	0,46429	
5	0,3336	0,7319	0,42392	0,47716	
4	0,2844	0,7111	0,41814	0,48960	
3	0,2269	0,6875	0,41237	0,50159	
2	0,1605	0,6611	0,40559	0,51314	
1	0,0850	0,6319	0,39815	0,52425	
0	0,0000	0,6000	0,39166	0,53492	0,11746
— 1	— 0,0954	0,5647	0,38417	0,54515	0,11842
— 2	— 0,2004	0,5277	0,37650	0,55494	0,11914
— 3	— 0,3169	0,4375	0,36861	0,56492	0,11972
— 4	— 0,4444	0,4444	0,36003	0,57319	0,11993
— 5	— 0,5835	0,3986	0,35119	0,58166	0,12022
— 6	— 0,7350	0,3500	0,34256	0,58967	0,11984
— 7	— 0,8624	0,2935	0,33352	0,59727	0,11991
— 8	— 1,0716	0,2444	0,32423	0,60441	0,11887
— 9	— 1,2656	0,1875	0,31493	0,61112	0,11854
— 10	— 1,4692	0,1277	0,30543	0,61738	0,11702
— 11	— 1,6869	0,0653	0,29570	0,62320	0,11688
— 12	— 1,9200	0,0000	0,28600	0,62858	0,11429

REFERENCES

[1] Розин Л. А., *Приближенний метод интегрирования уравнений нестационарного ламинарного пограничного слоя в несжимаемой жидкости*, П.М.М. т. 21, в. 5, 1957.

[2] Лойцянский Г. Л., *Ламинарный пограничный слой-Физматгиз*, Москва, 1962

[3] Goldstein S. and Rosenhead L., *Boundary-layer growth*, Proc. Camb. phil. Soc. 32, 1936.