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ON p — ADIC SPACES OF HENSEL

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While all p – adic fields are pairwise non isomorph the p – adic complete spaces are pairwise homeomorph.

1. *Definition of p – adic spaces* (cf. Hausdorff [1] p. 102). Let p, θ be any natural prime number and any real number satisfying $0<\theta<1$ respectively. For any $r \in Q$, $r \neq 0$ (Q denotes the set of rational numbers) let $r' = p(r)$ be the rational integer such that

$$r = p^{r'} s \tag{1}$$

where s is a rational number such that neither the numerator of s nor the denumerator of s is divisible by p; let

$$\left. \begin{aligned} r \in Q \longrightarrow \| r \| = \| r \|_{\theta p} &= \theta^{-r'} \\ \| 0 \| = 0. & \end{aligned} \right\} \tag{2}$$

The function (2) is a norm in Q, i. e. the function is defined in Q and has the following properties :

1. $\|x\| \geq 0$;
2. $\|x\| = 0 \iff r = 0$;
3. $\|x+y\| \leq \|x\| + \|y\|$ and still more 3' $\|x+y\| \leq \sup\{\|x\|, \|y\|\}$;
4. $\|xy\| = \|x\| \cdot \|y\|$.

2. *The space* $Q(\theta, p)$. Let $Q(\theta, p)$ be the space of rational numbers defined by means of the distance

$$\rho(x, y) = \rho_{\theta_p}(x, y) = \|x-y\|_{\theta_p} = \theta^{-p(x-y)}$$

3. *The space* $Q(\theta, p)$ *is dense in itself.*

As a matter of fact, every number $q \in Q$ is a point of accumulation of Q because e. g. the rational numbers $q - p^n = q_n$ converge to q:

$$\rho(q, q_n) = \|q - q_n\|_{\theta_p} = \|p^n\|_{\theta_p} = \theta^{-n} \rightarrow 0 \text{ as } n \rightarrow \infty .$$

On the other hand, if p' is any prime number $\neq p$, then

$$\rho_{\theta_{p'}}(q, q_n) = \| p^n \|_{\theta_{p'}} = \theta^0 = 1$$

because p^n is not divisible by p'. Consequently, the set $E_p = \{p, p^2, p^3, \cdots\}$ has 0 as limit point in the space $Q(p^{-1}, p)$ and has no limit point in the space $Q(\theta, p')$. Therefore the spaces $Q(\theta, p)$, $Q(\theta p')$ are not isomorphic by means of the identity mapping. However they are isomorphic :

4. Theorem. *For any pair p, q of prime numbers and for any $r \in Q \setminus \{0\}$ let $T_{pq}(r)$ be the number obtained from r by transposition of factors p, q. In other words, for every $r \in Q$ we have the mapping*

(1) $f_r : P \to D$ *such that*

(2) $r = \prod\limits_{p \in P} p^{f_r(p)}$; *$P$ denotes the set of all the prime numbers; D is the set of integers. Then*

(3) $$T_{pq}(r) := p^{f_r(q)}\, q^{f_r(p)}\, \Pi_x{}^{f_r(x)}, \quad (x \in P \setminus \{p, q\}),$$
$$T_{pq}(o) = o$$

is an isometry from $Q(\theta, p')$ onto $Q(\theta, q)$:

(4) $$\| r \|_{\theta_p} = \| T_{pq}\, r \|_{\theta_q}.$$

As a matter of fact, we have the following transformations (we assume that $r \neq o$):

$$(4)_2 = \| T_{pq}\, r \| \theta_q = \| T_{pq} \prod_{x \in p} x^{f_r(x)} \| \theta_p =$$

$$= \| p^{f_r(q)} q^{f_r(p)} \prod_{x \in P \setminus \{f, q\}} x^{f_r(x)} \|_{\theta q} = \theta^{-f_r(p)} = \| \prod_{x \in P} x^{f_r(x)} \|_{\theta p} = (4)_1 .$$

Hence, (4) holds.

5. *Equivalence of the norms* $\| r \|_{\theta_p}$, $\| r \|_{p^{-1} p}$. These two norms are equivalent in the sense that they produce C-homeomorphic spaces

$$Q(\theta, p), \quad Q(\overset{-1}{p}, p)$$

5.1 We define: A metric space (M, ρ) is *C-homeomorph* to a metric space (M', ρ') symbolically $(M, \rho) \approx_c (M', \rho')$

(1) if there exists a homeomorphism f from M onto M' by means of which the fundamental or Cauchy-sequences are mutually associated: if $x_1, x_2 \cdots$ is a Cauchy sequence in (M, ρ), then $fx_1, fx_2, \cdots$ is a Cauchy sequence in (M', ρ').

Then we have

(2) $$Q(\theta, p) \approx_c Q(\overset{-1}{p}, p),$$
$$Q(\theta, q) \approx_c Q(\overset{-1}{q}, q).$$

Since $Q(\theta, p) \overset{isom}{=\!=} Q(\theta, q)$, the relations (2) yield

(4) $$Q(\overset{-1}{p}, p) \approx_c Q(\overset{-1}{q}, q)$$

Such a special homeomorphy between the spaces $\overset{-1}{Q}(p, p)$, $(\overset{-1}{Q}(q, q)$ is the transformation (3) in the section 4.

6. *Hensel spaces. Hensel spaces are defined as metrical completions* $\overline{\overset{-1}{Q}(p, p)}$ of the spaces $\overset{-1}{Q}(p, p)$, p running through P.

For another $q \in Q$ we have the spaces $\overset{-1}{Q}(q, q)$, $Q(0, q)$. Then the continuous mapping

$$x \in \overline{\overset{-1}{Q}(p, p)} \longrightarrow fx \in \overline{\overset{-1}{Q}(q, q)}$$

satisfying $f \mid \overset{-1}{Q}(p, q) = T_{pq} \mid \overset{-1}{Q}(p, p)$

is a requested homeomorphism between the Hensel spaces $\overline{\overset{-1}{Q}(p, p)}$, $\overline{\overset{-1}{Q}(q, q)}$.

6.1 Consequently, up to isomorphism *there exists a single Hensel's complete space H over the field of rational numbers.* This space is not homeomorph to the space R of real numbers because dim $R = 1$ and dim $H = 0$. The equality dim $H = 0$ is implied by the special non archimedean condition 3' of the norm of spaces $\overset{-1}{Q}(p, p)$; the same condition holds also for the completion $\overline{\overset{-1}{Q}(p, p)}$ i. e. for the space H.

6.2 The foregoing *topological* poorness of Hensel's spaces is to be confronted with the *algebraic* fact that *the algebraic complete fields*

$$\overline{Q_p(+, \ )} \qquad (p \in P)$$

are pairwise non isomorphic (Mac Duffee p. 199/201, theorems 91.2, 91,4).

7. Remark. The fact that the metrical spaces $\overset{-1}{Q}(p, p)$, $\overset{-1}{Q}(q, q)$ are homeomorph is a special case of a theorem by W. Sierpiński that *every metrical space which is dense-in-itself is homeomorphic to the ordered space of rational numbers*; [v. Sierpiński 6]; also [3] Kuratowski I p. 175]; but the accent in the present situation is that the spaces are C-homeomorphic involving the isomorphism of the corresponding complete closures! E. g. the spaces $\overset{-1}{Q}(p, p)$, $(Q, <)$ are homeomorph but are not C-homeomorph.

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