PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 5 (19), 1965, p. 85—88.

REMARKS ON THE CAUCHY FUNCTIONAL EQUATION

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(Communicated May 22, 1964)

In this Note $R = \{x, y, \ldots\}$ denotes the set of all real numbers and $R_0 = \{x \mid x \in R,\ x \neq 0\}$. A function $f : R \to R$ is said to satisfy the Cauchy functional equation if

$$f(x+y) = f(x) + f(y) \tag{1}$$

holds for all $x, y, \in R$. A function $f : R \to R$ is termed a derivative on R if f satisfies the functional equation (1) and

$$f(xy) = x f(y) + y f(x) \tag{2}$$

holds for all $x, y \in R$.[1]

In [1] we have proved the following theorem:

Theorem I. Suppose that f and $g \neq 0$ are two solutions of (1) and that

$$g(x) = P(x)\, f(1/x) \tag{3}$$

holds for all $x \in R_0$ where P is a continuous function on R subject to the unessential condition $P(1) = 1$.

Then

$$f(x) + g(x) = 2 f(1)\, x \qquad (x \in R) \tag{4}$$

and the function

$$F(x) = f(x) - x f(1)$$

is a derivative on R.

This theorem in the case $f = g$ implies that a function f which satisfies (1) and $f(x) = x^2 f(1/x)$ for $x \in R_0$ is a continuous function. This solves the problem № 638 in The New Scottish book given by Prof. Israel Halperin in 1963. (See. Coll. Math. XI_1 (1963) p. 140), by which this work was also motivated. It is the object of this Note to generalize Theorem I. We get also another proof for this Theorem. We have

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[1] For the existence of a nontrivial derivatives on R see [2] pp. 120—131.

Theorem 1. *Suppose that f and $g \neq 0$ are two solutions of the Cauchy functional equation* (1) *and that there is an integer $n \neq 0, -1$ and a continuouos function $P : R_0 \to R$, subject to an unessential condition $P(1) = 1$, such that*

$$g(x) = P(x) f(1/x^n) \tag{5}$$

holds for all $x \in R_0$.

Then

$$P(x) = x^{n+1}, \; n f(x) + g(x) = (n+1) f(1) x \tag{6}$$

holds for all $x \in R$ and the function

$$F(x) = f(x) - x f(1)$$

is a derivative on R.

Conversely: If F is a derivative on R, $f(x) = ax + F(x)$ with a constant a $a \in R$, $g(x) = a(n+1)x - n f(x)$ and $P(x) = x^{n+1}$, then f and g are solutions of the Cauchy functional equation (1) *and* (5) *holds for every $x \in R_0$.*

Corollary 1. *Suppose that a function $f : R \to R$ satisfies the Cauchy functional equation* (1) *and that*

$$a f(x) = x^{n+1} f(1/x^n)$$

holds for all $x \in R_0$, where $n \neq 0, -1$ is an integer and $a \neq n$ a constant. Then $f(x) = x f(1)$ holds for all $x \in R$.

Proof of Theorem 1. We devide the proof in several steps.

I. The function P is of the form: $P(x) = x^{n+1}$ $(x \in R_0)$. In order to prove this we replace x by $r\,x\,(r \neq o)$ in (5) where from now an r denotes a rational number. We get

$$r\,g(x) = P(rx) \frac{1}{r^n} f\left(\frac{1}{x^n}\right)$$

which together with (5) leads to:

$$\left[\frac{P(rx)}{r^{n+1}} - P(x)\right] f\left(\frac{1}{x^n}\right) = 0. \tag{7}$$

If we multiply (7) with $P(x)$ and use (5) we get

$$\left[\frac{P(rx)}{r^{n+1}} - P(x)\right] g(x) = 0. \tag{8}$$

Now $g \neq 0$ implies the existence of $x_0 \neq 0$ such that $g(x_0) \neq 0$. But then (8) implies:

$$P(rx_0) = r^{n+1} P(x_0),$$

for all rational numbers $r \neq 0$. Using the continuity of P we find

$$P(x\,x_0) = x^{n+1} P(x_0) \tag{9}$$

for any $x \in R_0$. Replacing x with x/x_0 in (9) we get

$$P(x) = x^{n+1} \frac{P(x_0)}{x_0^{n+1}}$$

which because of $P(1)=1$ implies:

(10) $$P(x)=x^{n+1}$$

for any $x \in R_0$. Hence (10) and (5) imply:

(11) $$g(x)=x^{n+1} f(1/x^n) \qquad (x \in R_0).$$

If we set

(12) $$G(x)=g(x)-x\,g(1) \quad \text{and} \quad F(x)=f(x)-x\,f(1)$$

then (11) implies $f(1)=g(1)$ so that (12) and (11) lead to

(13) $$G(x)=x^{n+1}\, F(1/x^n) \quad (x \in R_0).$$

The functions F and G are solutions of the Cauchy functional equation (1), they satisfy (13) and $F(r)=G(r)=0$ for every rational number r.

II In order to prove Theorem 1 it is sufficient to prove that F is a derivative. Indeed if F is a derivative then $F(x)=-x^2\, F(1/x)$ and $F(y^k)=k\, y^{k-1}\, F(y)$ for any natural number k together with (13) leads to

(14) $$G(x)=-n\; F(x).$$

This is obvious if $n<0$. In the case $m=-n<1$ from (13) we get $x^{m-1}\, G(x)= = F(x^m)=m\, x^{m-1}\, F(x)$, i.e. $G(x)=m\, F(x)$ which implies (14).

From (14) Theorem 1 immediately follows. Now to prove that F is a derivative it is sufficient to prove that

(15) $$F(x^2)=2x\, F(x)$$

holds on R. Indeed if in (15) we replace x by $x+y$ we get:

$$F(x^2+2xy+y^2)=2(x+y)\; F(x+y)$$

which because of $F(x^2)=2x\; F(x)$ and $F(y^2)=2y\; F(y)$ implies $F(xy)=x\, F(y)+ + y\, F(x)$. Thus we have to prove the relation (15) only. In order to do this the case of positive n and of negative n are to be distinguished.

III Suppose that $m=-n \geqslant 2$ is a natural number.

Then (13) becomes:

$$F(x^m)=x^{m-1} \quad G(x).$$

Replacing x with $x+r$ in this relation, using the fact that $F(r^m)=0$ and $m \geqslant 2$ we get

$$F\left[m\, r^{m-1}\, x+\frac{m(m-1)}{2}\, r^{m-2}\, x^2+\cdots\right]=\left[r^{m-1}+(m-1)\, r^{m-2}\, x+\cdots\right] G(x), \text{ i. e.}$$

(16) $$m\, r^{m-1}\, F(x)+\frac{m(m-1)}{2}\, r^{m-2}\, F(x^2)+\cdots=\left[r^{m-1}+(m-1)\, r^{m-2}\, x+\cdots\right] G(x).$$

The equation (16) is an equation between two polinomials of order m—1 in r. From here by comparing coefficients of r^{m-1} and r^{m-2} we get:

$$m\, F(x)=G(x) \quad \text{and} \quad \frac{m(m-1)}{2}\, F(x^2)=(m-1)\;\, x\;\, G(x)$$

which implies $F(x^2)=2x\ F(x)$. Thus in the case of $n \in \{-2, -3, \cdot \cdot 4\}$ (15) is proved.

IV Suppose that n is a natural number. For an irrational number x we apply the function G on the identity:

$$\frac{1}{x-r} - \frac{1}{x+r} = 2r\frac{1}{x^2-r^2} \tag{17}$$

We get:

$$G\left(\frac{1}{x-r}\right) - G\left(\frac{1}{x+r}\right) = 2\ r\ G\left(\frac{1}{x^2-r^2}\right)$$

which because of (13) becomes:

$$\frac{1}{(x-r)^{n+1}} F[(x-r)^n] - \frac{1}{(x+r)^{n+1}} F[(x+r)^n] = 2r\frac{1}{(x^2-r^2)^{n+1}} F[(x^2-r^2)^n]\ .$$

We have therefore:

$$(x+r)^{n+1}\ F[(x-r)^n] - (x-r)^{n+1}\ F[(x+r)^n] = 2\ r\ F[(x^2-r^2)^n]\ , \tag{18}$$

i. e.

$$[r^{n+1} + (n+1)\ r^n x + \cdot \cdot \cdot]\ F\left[n(-r)^{n-1}x + \frac{n(n-1)}{2}(-r)^{9-2}x^2 + \cdot \cdot\right]$$

$$-\ [(-r)^{n+1} + (n+1)\ (-r)^n\ x + \cdot \cdot \cdot]\ F\left[n\ r^{n-1}\ x + \frac{n(n-1)}{2}\ r^{n-2}x^2 + \cdot \cdot\right]$$

$$=\ 2\,r\,F[\,n\,(-r^2)^{n-1}\ x^2 + \cdot \cdot \cdot\,].$$

Since this equation is an equation of two polinomials in r, we may compare the corresponding coefficients. Comparing coefficients of r^{2n-1} we get

$$\left[(-1)^{n-2}\ \frac{n\,(n-1)}{2}\ F(x^2) + (-1)^{n-1}(n+1)\ \ n\,x\ \ F(x)\right] -$$

$$-\left[(-1)^{n+1}\frac{n(n-1)}{2}\ F(x^2) + (-1)^n\,(n+1)n\ \ x\ \ F(x)\right] = 2(-1)^{n-1}\ n\ F(x^2)$$

from which follows $F(x^2) = 2\ x\ F(x)$.

Q.E.D.

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[1] S. Kurepa, *The Cauchy functional equation and scalar product in vector spaces* Glasnik mat. fiz. astr. **19** (1964), 23—36.

[2] O. Zariski and P. Samuel, *Cummutative algebra.* D. von Nostrand Company Inc. 1958.