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GENERALIZATION OF BONNET'S FORMULA FOR A SUBSPACE OF A GENERALIZED RIEMANN SPACE

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1. Introduction:

A space of coordinates y^α $(\alpha=1, 2, \ldots, m)$ with which is associated a non-symmetric tensor $a_{\alpha\beta}$ is called a generalised Riemann space X_m [1, 2]. Using bar and hook to denote symmetric and skew-symmetric parts respectively of a quantity, we have

$$a_{\underline{\alpha\beta}}=\frac{1}{2}(a_{\alpha\beta}+a_{\beta\alpha})$$

$$a_{\underset{\smile}{\alpha\beta}}=\frac{1}{2}(a_{\alpha\beta}-a_{\beta\alpha}).$$

By assuming $|a_{\alpha\beta}|\neq 0$, the quantities $a^{\underline{\alpha\beta}}$ are defined by

$$a^{\underline{\alpha\beta}}\,a_{\underline{\alpha\gamma}}=\delta^\beta_\gamma.$$

For such a space Eisenhart has obtained the affine connections $\Delta^\gamma_{\alpha\beta}$ defined by:

$$\Delta^\gamma_{\alpha\beta}=a^{\underline{\gamma\delta}}\,\Delta_{\alpha\beta\delta}$$

$$\Delta_{\alpha\beta\delta}=\frac{1}{2}\left(\frac{\partial a_{\alpha\delta}}{\partial y^\beta}+\frac{\partial a_{\delta\beta}}{\partial y^\alpha}-\frac{\partial a_{\alpha\beta}}{\partial y^\delta}\right).$$

The object of this note is to define geodesic torsion of a curve in a subspace of a generalised Riemann space and to obtain an expression for the same.

2. Subspace of X_m:

Consider a generalised Riemann space X_n of coordinates x^i $(i=1, 2, \ldots, n)$ imbedded in X_m as defined in § 1. Let g_{ij}, Δ^h_{ij} be the non-symmetric tensor and the affine connection respectively in X_n.

Then [1, 6],

$$g_{\underline{ij}}=a_{\underline{\alpha\beta}}\,y^\alpha_{,i}\,y^\beta_{,j}$$

$$a_{\alpha\beta;\,k}=0, \tag{2.1}$$

$$g_{ij;\,k}=0$$

where comma followed by an index denotes covariant derivative with respect to the y's (or x's) with that index and the connection $\Delta^{\gamma}_{\alpha\beta}$ (or Δ^{h}_{ij}) and semi-colon followed by an index denotes the tensor derivative with respect to x with that index. Also,

$$y^{\alpha}_{;ij} = \sum_{\nu} \Omega_{\nu|ij} N^{\alpha}_{\nu|} \tag{2.2}$$

$N^{\alpha}_{\nu|}$ being a set of $(m-n)$ mutually orthogonal unit vectors in X_m defined at points of X_n and normal to X_n and $\Omega_{\nu|ij}$ is a covariant tensor in X_n. The magnitude of a vector in X_n and the angle between two vectors are defined in the usual manner with respect to the tensor $a_{\alpha\beta}$.

The tensor derivative of $N^{\alpha}_{\nu|}$ is given by:

$$N^{\alpha}_{\nu|;i} = -g^{lj}\Omega_{\nu|li}\, y^{\alpha}_{,j} + \sum_{\mu} B_{\mu\nu|i} N^{\alpha}_{\mu|}, \tag{2.3}$$

where for each μ and ν the quantities $B_{\mu\nu|i}$ are the covariant components of a vector, subject to the conditions $B_{\mu\nu|i} + B_{\nu\mu|i} = 0$, $B_{\mu\mu|i} = 0$.

The Frenet's formulae for a curve C in X_n corresponding to an ortho-normal basis $\underset{(\rho)}{\lambda^{\alpha}}$ chosen suitably from the tangent vector t^{a} of C are given by [5]:

$$\frac{D\underset{(\rho)}{\lambda^{\alpha}}}{Ds} = -\varkappa_{\rho-1} \underset{(\rho-1)}{\lambda^{\alpha}} + \varkappa_{\rho} \underset{(\rho+1)}{\lambda^{\alpha}}, \tag{2.4}$$

where $\frac{D}{Ds}$ denotes intrinsic derivative in the direction of C and the $\varkappa$'s and λ's are given by:

$$\underset{(1)}{\lambda^{\alpha}} = \xi^{\alpha}_{(1)}, \quad \underset{(1)}{\lambda^{\alpha}} = \frac{1}{\sqrt{D_{\rho\lambda} D_{\rho}}} \begin{vmatrix} (1,1)\cdots(1,\rho-1) & \xi^{\alpha}_{(1)} \\ (2,1)\cdots(2,\rho-1) & \xi^{\alpha}_{(2)} \\ \vdots & \\ (\rho,1)\cdots(\rho,\rho-1) & \xi^{\alpha}_{(\nu)} \end{vmatrix} \tag{2.5}$$

$$\xi^{\alpha}_{(1)} = t^{\alpha}_{(1)} \qquad \xi^{\alpha}_{(\rho)} = \frac{D\xi^{\alpha}_{(\rho-1)}}{Ds} \tag{2.6}$$

$$D_0 = 1, \; D_{\rho} = \begin{vmatrix} (1,1) & \cdots & (1,\rho) \\ (2,1) & \cdots & (2,\rho) \\ \vdots & & \\ (\rho,1) & \cdots & (\rho,\rho) \end{vmatrix} \tag{2.7}$$

$$(\rho, \sigma) = a_{\alpha\beta}\, \xi^{\alpha}_{(\rho)} \xi^{\beta}_{(\sigma)}$$

$$\varkappa_{\rho} = \frac{\sqrt{D_{\rho-1} D_{\rho+1}}}{D_{\rho}} \qquad \varkappa_0 = 0 = \varkappa_m$$

3. Geodesic Torsion of a curve in X_n:

Consider a curve $x^{i} = x^{i}(s)$ in X_n and let $t^{i}_{(1)}$ be the components of the unit tangent to C in the x's. Then the components $t^{\alpha}_{(1)}$ in X_m are given by:

$$t^{\alpha}_{(1)} = y^{\alpha}_{,i}\, t^{i}_{(1)} \tag{3.1}$$

Taking the intrinsic derivatives of both sides along C, we get

(3.2) $$\frac{Dt^{\alpha}_{(1)}}{Ds}=y^{\alpha}_{;ij}t^{i}_{(1)}t^{j}_{(1)}+y^{\alpha}_{,i}t^{i}_{(1);j}t^{j}_{(1)}=y^{\alpha}_{,i}t^{i}_{(1);j}t^{j}_{(1)}+\sum_{\nu}\Omega_{\nu|ij}N^{\alpha}_{\nu|}t^{i}_{(1)}t^{j}_{(1)}$$

which can be written as:

(3.2)′ $$t^{\alpha}_{(2)}=y^{\alpha}_{,i}t^{i}_{(2)}+\sum_{\nu}\underset{(1)}{\varkappa_{\nu|}}N^{\alpha}_{\nu|}$$

where we have used:

$$t^{\alpha}_{(p)}=\frac{Dt^{\alpha}_{(p-1)}}{Ds}\qquad p\geqslant 2$$

(3.4) $$t_{(p)}=t^{i}_{(p-1);j}t^{j}_{(p-1)}-\sum_{\nu}\underset{(p-2)}{\varkappa_{\nu|}}\Omega_{\nu|jk}g^{kl}t^{j}_{(p-2)}$$

(3.5) $$\underset{(p)}{\varkappa_{\nu|}}=\frac{D\underset{(p-1)}{\varkappa_{\nu|}}}{Ds}+\sum_{\mu}\underset{(p-1)}{\varkappa_{\mu|}}B_{\mu\nu|j}t^{j}_{(p-1)}+\Omega_{\nu|ij}t^{i}_{(p)}t^{j}_{(p-1)}$$

$$\underset{(0)}{\varkappa_{\nu|}}=0\,.$$

Taking the intrinsic derivative of (3.2)′ along C we get

$$\frac{Dt^{\alpha}_{(2)}}{Ds}=y^{\alpha}_{;ij}t^{i}_{(2)}t^{j}_{(1)}+t^{i}_{(2);j}y^{\alpha}_{,i}y^{\alpha}_{,j}+\sum_{\nu}\frac{D\underset{(1)}{\varkappa_{\nu|}}}{Ds}N^{\alpha}_{\nu|}+\sum_{\nu}\underset{(1)}{\varkappa_{\nu|}}N^{\alpha}_{\nu|;j}t^{j}_{(1)}$$

Therefore, from (2.2), (3.4) and (3.5) this can be written as

(3.6) $$t^{\alpha}_{(3)}=y^{\alpha}_{,i}t^{i}_{(2)}+\sum\varkappa_{\nu|}N^{\alpha}_{\nu|}\,.$$

Proceeding in a similar manner we will get:

(3.7) $$t^{\alpha}_{(p)}=y^{\alpha}_{,i}t^{i}_{(p)}+\sum_{\nu}\underset{(p-1)}{\varkappa_{\nu|}}N^{\alpha}_{\nu|}\qquad\text{for } p=1,2,\ \dots,\ m.$$

The vectors $t^{i}_{(p)}$ are linearly independent and we may obtain through linear combinations of them, m mutually orthogonal unit vectors $\underset{(p)}{\lambda}$ $(p=1,2,\ \dots,\ m)$.

Let

(3.8) $$a_{\alpha\beta}t^{\alpha}_{(p)}t^{\beta}_{(q)}=(p,q)$$

and D_p denote the determinant $|(r,q)|$ $(r,q=1,2,\ \dots,\ p)$. Consider the vector with components $\underset{(p)}{\lambda^{\alpha}}$ which are given by:

(3.9) $$\underset{(p)}{\lambda^{\alpha}}=\sqrt{\frac{D_p}{D_{p-1}}}\,t^{\alpha}_{(p)}B^{r}_{p},\qquad r=1,2,\ \dots,\ p$$

where B^{r}_{p} is the cofactor of (r,p) in D_p divided by $|D_p|$. The vectors $\underset{(p)}{\lambda^{\alpha}}$ as defined by (3.9) for $(p=1,2,\ \dots,\ m)$ form an orthogonal ennuple of unit vectors [3]. Using (3.7) we get:

(3.10) $$(p,q)=g_{ij}t^{i}_{(p)}t^{j}_{(q)}+\sum_{\nu}\underset{(p-1)}{\varkappa_{\nu|}}\underset{(q-1)}{\varkappa_{\nu|}}$$

and from (3.9) we get:

(3.11) $$\underset{(p)}{\lambda^{\alpha}}=\sqrt{\frac{D_p}{D_{p-1}}}B^{r}_{p}\left(y^{\alpha}_{,i}t^{i}_{(r)}+\sum_{\nu}\underset{(r-1)}{\varkappa_{\nu|}}N^{\alpha}_{\nu|}\right)=E^{i}_{(p)}y^{\alpha}_{,i}+\sum N^{\alpha}_{\nu|}\underset{(p)}{F_{\nu|}}$$

where

$$E^i_{(p)}=\sqrt{\frac{D_p}{D_{p-1}}}\,B^r_p\,t^i_{(r)}\,, \tag{3.12}$$

$$\underset{(p)}{F_{\nu|}} = \underset{(p-1)}{\varkappa_{\nu|}}\,B^r_p\sqrt{\frac{D_p}{D_{p-1}}}\,, \tag{3.13}$$

From (2.4) the Frenet's formulae for λ^α's become

$$\frac{D\lambda^\alpha_{(p)}}{Ds}=-\varkappa_{p-1}\lambda^\alpha_{(p-1)}+\varkappa_p\lambda^\alpha_{(p+1)} \tag{3.14}$$

Now let $\underset{p}{\theta_{\nu|}}$ be the angle between $\lambda_{(p)}$ and $N_{\nu|}$ so that

$$\underline{a_{\alpha\beta}}\,N^\alpha_{\nu|}\,\lambda^\beta_{(p)}=\cos\underset{p}{\theta_{\nu|}}$$

Taking the intrinsic derivative of both sides along C

$$\underline{a_{\alpha\beta}}\,N^\alpha_{\nu|;i}\,t^i_{(1)}\,\lambda^\beta_{(p)}+\underline{a_{\alpha\beta}}\,N^\alpha_{\nu|}\,\frac{D\lambda^\beta_{(p)}}{Ds}=-\sin\underset{p}{\theta_{\nu|}}\,\frac{D\underset{p}{\theta_{\nu|}}}{Ds}$$

using (3.14)

$$-\varkappa_{p-1}\cos\underset{p-1}{\theta_{\nu|}}+\varkappa_p\cos\underset{p+1}{\theta_{\nu|}}+\sin\underset{p}{\theta_{\nu|}}\,\frac{D\underset{p}{\theta_{\nu|}}}{Ds}=H_{\nu|\beta}\,\lambda^\beta_{(p)}\,, \tag{3.15}$$

where

$$H_{\nu|\beta}=\underline{a_{\alpha\beta}}\,N^\alpha_{\nu|;i}\,t^i_{(1)}$$

$$=\underline{a_{\alpha\beta}}\left[-\Omega_{\nu|ik}\,\underline{g^{kj}}\,y^\alpha_{,j}\,t^i_{(1)}+\sum_\mu B_{\mu\nu|i}\,N^\alpha_{\mu|}\,t^i_{(1)}\right] \tag{3.16}$$

From (3.11) it follows that

$$H_{\nu|\beta}\,\lambda^\beta_{(p)}=\left[-E^i_{(p)}\Omega_{\nu|il}+\sum_\mu \underset{(p)}{F_{\mu|}}\,B_{\mu\nu|l}\right]t^l_{(1)}$$

and (3.15) takes the form

$$-\varkappa_{p-1}\cos\underset{p-1}{\theta_{\nu|}}+\varkappa_p\cos\underset{p+1}{\theta_{\nu|}}+\sin\underset{p}{\theta_{\nu|}}\,\frac{D\underset{p}{\theta_{\nu|}}}{Ds}$$

$$=\left[-E^i_{(p)}\,\Omega_{\nu|il}+\sum_\mu \underset{(p)}{F_{\mu|}}\,B_{\mu\nu|l}\right]t^j_{(1)} \tag{3.17}$$

To express the formula (3.17) in a different form we solve the m equations:

$$\left.\begin{aligned}\underline{a_{\alpha\beta}}\,\lambda^\alpha_{(p)}\,\lambda^\beta_{(q)}&=0\\ \underline{a_{\alpha\beta}}\,\lambda^\alpha_{(p)}\,\lambda^\beta_{(p)}&=1\end{aligned}\right\}\quad q=1,2,\ldots m\quad (p\neq q) \tag{3.18}$$

for the m quantities $\underset{(p)}{\lambda_\alpha}$. If $\underset{(p)}{\Lambda^\beta}$ denotes the cofactor of $\underset{(p)}{\lambda^\beta}$ in the determinant $\left|\underset{(p)}{\lambda_\beta}\right|$ the solution of (3.18) are given by:

$$\underset{(p)}{\lambda^\alpha}=\frac{\underset{(p)}{\Lambda^\alpha}}{\left|\underset{(p)}{\lambda_\alpha}\right|}=\frac{\underset{(p)}{\Lambda^\alpha}}{\sqrt{|a_{\alpha\beta}|}}=\underset{(p)}{\Lambda^\alpha}\,\frac{|q^\alpha_\beta|}{\sqrt{|g_{ij}|}} \tag{3.19}$$

where $q^{\alpha}_{\beta} = y^{\alpha}_{,i}$ when $\beta = i, \quad i = 1, \ldots, n$

$= N^{\alpha}_{\nu|}$ when $\beta = \nu, \quad \nu = n+1), \ldots, m.$

Hence the right hand side of (3.15) can be written as

$$\frac{1}{\sqrt{|g_{ij}|}} H_{\nu|\beta} \underset{(p)}{\dot{\Lambda}^{\beta}} \; |q^{\alpha}_{\gamma}|$$

$$\text{i.e.} \quad \frac{|q^{\alpha}_{\gamma}|}{\sqrt{|g_{ij}|}} (-1)^{p-1} \begin{vmatrix} H_{\nu|1} & \underset{(1)}{\lambda_1} & \cdots & \underset{(m)}{\lambda_1} \\ H_{\nu|2} & \underset{(1)}{\lambda_2} & & \underset{(m)}{\lambda_2} \\ \vdots & & & \\ H_{\nu|m} & \underset{(1)}{\lambda_m} & & \underset{(m)}{\lambda_m} \end{vmatrix}$$

$$\text{i.e.} \quad \frac{(-1)^{p-1}}{\sqrt{|g_{ij}|}} \begin{vmatrix} H_{\nu|\alpha} y^{\alpha}_{,1} & H_{\nu|\alpha} y^{\alpha}_{,2} & \cdots & H_{\nu|\alpha} y^{\alpha}_{,n} & H_{\nu|\alpha} N^{\alpha}_{n+1} & \cdots & H_{\nu|\alpha} N^{\alpha}_{m|} \\ \underset{(1)}{\lambda_{\alpha}} y^{\alpha}_{,1} & \underset{(2)}{\lambda_{\alpha}} y^{\alpha}_{,2} & & & & & \underset{(1)}{\lambda_{\alpha}} N^{\alpha}_{m|} \\ \vdots & & & & & & \\ \underset{(m)}{\lambda_{\alpha}} y^{\alpha}_{,1} & \underset{(m)}{\lambda_{\alpha}} y^{\alpha}_{,2} & & \underset{(m)}{\lambda_{\alpha}} y^{\alpha}_{,n} & \underset{(m)}{\lambda_{\alpha}} N^{\alpha}_{n+1} & \cdots & \underset{(m)}{\lambda_{\alpha}} N^{\alpha}_{m|} \end{vmatrix}$$

Also we have,

$$H_{\nu|\alpha} y^{\alpha}_{,i} = -\Omega_{\nu|il} \frac{dx^l}{ds},$$

$$H_{\nu|\alpha} N^{\alpha}_{\sigma_1} = B_{\sigma\nu|i} \frac{dx^i}{ds},$$

$$\underset{(p)}{\lambda_{\alpha}} y^{\alpha}_{,i} = \underset{(p)}{E_i}$$

$$\underset{(p)}{\lambda_{\alpha}} N^{\alpha}_{y|} = \underset{(p)}{F_{\nu|}} = \cos \theta_{\nu_1|}.$$

Using these, we can write (3.17) in the form

$$(3.20) \quad -\varkappa_{p-1} \cos \underset{p-1}{\theta_{\nu|}} + \varkappa_p \cos \underset{p+1}{\theta_{\nu|}} + \sin \underset{p}{\theta_{\nu|}} \frac{D \underset{p}{\theta_{\nu|}}}{Ds}$$

$$= \frac{(-1)^p}{\sqrt{|g_{ij}|}\,(Ds)^2} \begin{vmatrix} \Omega_{\nu|1l}\, dx^l & \Omega_{\nu|2l}\, dx^l & \cdots & \Omega_{\nu|nl}\, dx^l & B_{\underset{\nu}{n+1}|l}\, dx^l & \cdots & B_{\mu\nu|l}\, dx^l \\ g_{1l} dx^l & g_{2l}\, dx^l & \cdots & g_{nl}\, dx^l & 0 & 0 \cdots & 0 \\ \underset{(2)}{E_1} & \underset{(2)}{E_2} & \cdots & \underset{(2)}{E_n} & \cos \underset{2}{\theta_{n+1|}} & \cdots & \cos \underset{2}{\theta_{m}} \\ \vdots & & & & & & \\ \underset{(m)}{E_1} & \underset{(m)}{E_2} & \cdots & \underset{(m)}{E_n} & \cos \underset{m}{\theta_{n+1|}} & \cdots & \cos \underset{m}{\theta_{m|}} \end{vmatrix}$$

4. Particular cases:

I. If $m=n+1$, $B_{\mu\nu|}=0$ we get

$$(4.1)\qquad -\varkappa_{p-1}\cos\underset{p-1}{\theta_{\nu|}}+\varkappa_p\cos\underset{p+1}{\theta_{\nu|}}+\sin\underset{p}{\theta_{\nu|}}\frac{D\underset{p}{\theta_{\nu|}}}{Ds}$$

$$=\frac{(-1)^p}{\sqrt{\underline{g_{ij}}}\,(Ds)^2}\begin{vmatrix} \Omega_{1l}\,dx^l & \Omega_{2l}\,dx^l & \cdots & \Omega_{nl}\,dx^l & \cdots & 0 \\ \underline{g_{1l}}\,dx^l & \underline{g_{2l}}\,dx^l & \cdots & \underline{g_{nl}}\,dx^l & \cdots & 0 \\ \underset{(2)}{E_1} & \underset{(2)}{E_2} & \cdots & \underset{(2)}{E_n} & \cdots & \cos\theta_2 \\ \vdots & & & & & \\ \underset{(n+1)}{E_1} & \underset{(n+1)}{E_2} & \cdots & \underset{(n+1)}{E_n} & \cdots & \cos\theta_{n+1} \end{vmatrix}$$

which is the generalization of the known result of Kawaguchi and Hosokawa [4].

II. If $\underline{g_{ij}}=0$,

we get the case for a Riemannian subspace in a Riemannian space [6].

III. When $\underset{\vee}{g_{ij}}=0$, for a surface in a Euclidean 3-space we get

$$\theta_1=\frac{\pi}{2},\quad \theta_2=\omega,\quad \theta_3=\frac{\pi}{2}-\omega.$$

so that for $p=2$, (4.1) yields

$$(4.2)\qquad \varkappa_2+\frac{D\omega}{Ds}=\frac{1}{\sqrt{|\underline{g_{ij}}|}\,(Ds)^2}\begin{vmatrix} \Omega_{1l}\,dx^l & \Omega_{2l}\,dx^l \\ g_{1l}\,dx^l & g_{2l}\,dx^l \end{vmatrix},$$

which is Bonnet's formula for the geodesic torsion of a curve. Equations (3.20) are therefore generalisation of the Bonnet's formula for a curve in a subspace of a generalised Riemannian space.

REFERENCES

1. Eisenhart, L. P. (1951) *Generalized Riemann Spaces I.* Proc. Nat. Acad. of Sci. U.S.A. Vol 37, 311—315.

2. Eisenhart, L. P. (1952) *Generalized Riemann Spaces II.* Proc. Nat. Acad. of Sci. U.S.A Vol 38, 505—508.

3. Eisenhart, L. P. (1949) *Riemannian Geometry.*

4. Kawaguchi & Hosokawa (1933) *Über die geodätische Torsion in der Riemanschen Mannigfaltigkeit,* Tohuku Math. Jour. Vol 37, 340—

5. Shamihoke, A. C. (1960) *On Frenet's formulae and curvature in a Generalized Riemann Space.* Revue de la Faculté des Sciences de l'Université d'Istanbul, A 25, 9—16.

6. Singal, M. K. (1958) *Differential Geometry of Subspaces.*