PUBLICATION DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 4 (18), 1964, p. 77—80.

ON A GENERALIZATION OF THE FUNCTIONAL EQUATIONS OF PEXIDER

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(Presented December 6, 1963)

0. N. W. Pexider [2] has solved the functional equations

$$f(x+y)=g(x)+h(y),\ f(x+y)=g(x)h(y),$$

$$f(xy)=g(x)+h(y),\ f(xy)=g(x)h(y)$$

for continuous real functions.

E. Vincze [3] has treated the more general equation

$$f(x\cdot y)=g(x)h(y) \tag{1}$$

where the functions f, g and h are defined in a semigroup S and the values of these functions are elements of a group G possibly enlarged by a zero-element. He gave the general solution of (1) under the supposition that there exist $a, b\in S$ so that for any $u, v\in S$, the equations

$$a\cdot x=u \quad \text{and} \quad y\cdot b=v \tag{2}$$

have (at least one) solution x resp. y in S. ($a\cdot S=S\cdot b=S$.)

In the terminology of binary systems (cf. [1]), (1) can be formulated so that (f, g, h) is *a homotopism of the semigroup S into the group G.*

At a Conference on Functional Equations in Sárospatak (May 1962) he raised the question whether in case of *abelian* semigroups S the conditions of solvability of the equations (2) can be replaced by the weaker condition that (3) $S\cdot S=S$, i.e., *the product $x\cdot y$ assumes every value in S (at least once).*

(We will call such semigroups *transitive.*)

In this paper we show that the answer is positive, moreover if the values of the functions f, g, h lie in a *group G* (i.e., no zero-element is allowed), then also condition (3) can be left aside.

1. In fact, S being abelian, (1) involves

$$g(x)h(y)=f(x\cdot y)=f(y\cdot x)=g(y)h(x)$$

and by substituting a constant value $y=y_0$ in this equation we get

$$g(x)=dh(x)b^{-1} \tag{4}$$

where $d=g(y_0)$ and

$$b=h(y_0). \tag{5}$$

If we resubstitute (4) into (1) we get

$$f(x \cdot y) = dh(x) b^{-1} h(y) \tag{6}$$

and S being a semigroup

$$dh(x \cdot y) b^{-1} h(z) = f((x \cdot y) \cdot z) = f(x \cdot (y \cdot z)) = dh(x) b^{-1} h(y \cdot z).$$

If we put here again $z = y_0$, we get by (5) and by the group-property of G

$$h(x \cdot y) = h(x) b^{-1} h(y \cdot y_0)$$

or with $k(y) = b^{-1} h(y \cdot y_0)$:

$$h(x \cdot y) = h(x) k(y). \tag{7}$$

Again, $x \cdot y$ being commutative

$$h(x) k(y) = h(x \cdot y) = h(y \cdot x) = h(y) k(x)$$

and with $y = y_0$

$$h(x) = bk(x) c \tag{8}$$

where

$$c = k(y_0)^{-1} \tag{9}$$

If we put (8) back into (7) we get

$$bk(x \cdot y) c = bk(x) ck(y)$$

i.e.,

$$k(x \cdot y) = k(x) ck(y) c^{-1}. \tag{10}$$

We want to show that c is interchangeable with the values of k and thus

$$k(x \cdot y) = k(x) k(y) \tag{11}$$

i.e., *k is a homomorphism of S into G*. For this reason we make again use of $x \cdot y$ being associative: by (10)

$$k(x \cdot y) ck(z) c^{-1} = k((x \cdot y) \cdot z) = k(x \cdot (y \cdot z))$$
$$= k(x) ck(y \cdot z) c^{-1}$$

and again by (10)

$$k(x) ck(y) k(z) = k(x) ck(y) ck(z) c^{-1}$$

i.e.

$$k(z) c = ck(z) \tag{12}$$

and thus (11) follows from (10). (11) shows also that

$$k(x) k(y) = k(y) k(x)$$

i.e. the values of $k(t)$ are interchangeable.

(12) shows at the same time that (8) can be written in the form

$$h(x) = bck(x)$$

and by (4) with $a = dbc$

$$g(x) = ak(x) b^{-1}.$$

Thus finally (1), (11) and (12) give

$$f(x \cdot y) = ak(x) ck(y) = ak(x \cdot y) c$$

and if we replace ab^{-1} by a, bc by c and $bk(x) b^{-1}$ by $k(x)$ we have

Theorem 1. *The most general solutions of equation*

$$f(x\cdot y)=g(x)\,h(y)$$

among the functions f, g, h mapping an abelian semigroup S into a group G are given by

$$g(x)=ak(x),\quad h(y)=k(y)\,c,\quad f(x\cdot y)=ak(x\cdot y)\,c \tag{13}$$

where a and c are constants in G, and k is a homomorphism:

$$k(x\cdot y)=k(x)\,k(y).$$

In fact, (13) satisfies (1) with arbitrary constants a, b, c.

Corollary. If (f, g, h) is a homotopism of an abelian semigroup, S into a group G, then there exists a homomorphism k of S into G so that (13) *holds.*

2. Now let us alter the suppositions so that the *functions f, g and h map the transitive abelian semigroup S into a set G [0] consisting of a group G and an element 0 with the multiplication rule*

$$0a=a0=0.$$

(Of course, 0 has no inverse element.) — The steps similar to those in **1** will be described but shortly.

Departing from

$$g(x)\,h(y)=g(y)\,h(x)$$

we must first consider the case $h(y)\equiv 0$ which gives the solution

$$g(x)\ \text{arbitrary},\quad h(y)=0,\quad f(x\cdot y)=0. \tag{14}$$

But if $h(y)\not\equiv 0$ then there exists an y_0 so that $b=h(y_0)\neq 0$ and (4) and (6) remain valid. But

$$dh(x\cdot y)=dh(x)\,k(y)$$

implies

$$h(x\cdot y)=h(x)\,k(y) \tag{7}$$

only if $d\neq 0$. In the contrary case $d=0$ and by (4) $g(x)\equiv 0$ so that we have another solution (the counterpart of (14)):

$$g(x)=0,\quad h(y)\ \text{arbitrary},\quad f(x\cdot y)=0. \tag{15}$$

Otherwise we can march further till

$$h(x)\,k(y_0)=h(y_0)\,k(x)$$

but this implies (8) only if $k(y_0)\neq 0$. In the contrary case from the same equation (as $h(y_0)=b\neq 0$) $k(x)\equiv 0$ and from (7)

$$h(x\cdot y)\equiv 0$$

follows. By the transitivity supposition this is equivalent with $h(y)\equiv 0$ and so we get again the solution (14)*.

Further everything goes on the same lines as in **1** and we get yet the solution (13) and so we have the

Theorem 2. *The most general solutions of* (1) — *among the functions* f, g, h *mapping a transitive abelian semigroup* S *into* $G[0]$ *are* (13), (14), *and* (15).

The author is indebted to L. Fuchs for some valuable remarks.

* This is the only point where we made use of the transitivity of S. But here it cannot be left aside: $h(x\cdot y)\equiv 0$ in general does not imply $h(z)\equiv 0$. In fact, let us take instead of (6) $f(x\cdot y)=h(x)\,h(y)$ (to which (6) always can be reduced) and S a finite semigroup with 3 elements and the multiplication

	1	2	3
1	1	1	1
2	1	1	1
3	1	1	2

This multiplication is not transitive but commutative and associative, and

$$h(1)=h(2)=0,\quad h(3)=1;\quad f(1)=0,\quad f(2)=1$$

is a solution of $f(x\cdot y)=h(x)\,h(y)$ (in $G[0]$ multiplication is the ordinary product of numbers) with $h(x\cdot y)\equiv 0$ but $h(z)\not\equiv 0$. (This counter-example is essentially due to A. Rényi.)

Another counter-example found by L. Losonczi, where S is an infinite semigroup with ordinary multiplication as semigroup-operation, is the following: S is the semigroup of entires greater than 1 under ordinary multiplication. Now define $h(p)=1$ if p is a prime number and $h(n)=0$ if n has at least two (not necessarily different) prime factors. At the same time, let

$$f(n)=\begin{cases} 1 & \text{if } n \text{ has at most two prime factors,}\\ 0 & \text{else.}\end{cases}$$

These functions again satisfy

$$f(x\cdot y)=h(x)\,h(y)$$

(in $G[0]$ too multiplication is ordinary product of entires while $h(x\cdot y)\equiv 0$ but $h(z)\not\equiv 0$.

REFERENCES

[1] R. H. Bruck, *A Survey of Binary Systems.* Berlin-Göttingen-Heidelberg, 1958.

[2] H. W. Pexider, *Notiz über Funktionaltheoreme.* Monatsh. Math. Phys. 14, (1903), 293—301.

[3] E. Vincze, *Über eine Verallgemeinerung der Pexiderschen Funktionalgleichungen.* Studia Univ. Babes-Bolyai. 1962.