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SUMMATION OF CERTAIN TYPES OF SERIES

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I

Let $F(x)$ be a rational function having only real integral poles and such that

$$\lim_{x \to +\infty} |x^2 F(x)| < +\infty . \tag{1}$$

We consider the numerical series

$$\sum_{x=r}^{\infty} F(x), \tag{2}$$

where r is greater than the maximal pole of $F(x)$. The sum of such a series can be expressed in finite form by means of the Riemann ζ-function.

Indeed, let

$$\begin{aligned} F(x) = & \frac{C_{11}}{x-\alpha_1} + \frac{C_{12}}{(x-\alpha_1)^2} + \cdots + \frac{C_{1i_1}}{(x-\alpha_1)^{i_1}} \\ & + \cdots \\ & + \frac{C_{k1}}{x-\alpha_k} + \frac{C_{k2}}{(x-\alpha_k)^2} + \cdots + \frac{C_{ki_k}}{(x-\alpha_k)^{i_k}} \end{aligned} \tag{3}$$

be decomposition of $F(x)$ into partial fractions. Summing over x from r to ∞ we get

$$\begin{aligned} \sum_{x=r}^{\infty} F(x) = & \sum_{x=r}^{\infty} \left(\frac{C_{11}}{x-\alpha_1} + \frac{C_{21}}{x-\alpha_2} + \cdots + \frac{C_{k1}}{x-\alpha_k} \right) \\ & + C_{12} \sum_{x=r}^{\infty} \frac{1}{(x-\alpha_1)^2} + \cdots + C_{1i_1} \sum_{x=r}^{\infty} \frac{1}{(x-\alpha_1)^{i_1}} \\ & + \cdots \\ & + C_{k2} \sum_{x=r}^{\infty} \frac{1}{(x-\alpha_k)^2} + \cdots + C_{ki_k} \sum_{x=r}^{\infty} \frac{1}{(x-\alpha_k)^{i_k}} . \end{aligned} \tag{4}$$

It follows from (1) that $C_{11} + C_{21} + \cdots + C_{k1} = 0$ so that the first sum on the right-hand side of (4) reduces to a finite sum. By adding a finite number of terms each of the remaining sums of the right-hand member of (4) can be replaced by $\zeta(a)$ where a is some integer $\geqslant 2$.

If only even numbers appear as arguments of the ζ-function, then the sum (2) can be expressed in finite form by means of π since

$$\zeta(2s)=\frac{2^{2s-1}}{(2s)!}\pi^{2s}\,|B_{2s}|\,. \tag{5}$$

In last formula B_s are Bernoullian numbers defined by the expansion

$$\frac{t}{e^t-1}=\sum_{\nu=0}^{\infty}B_\nu\frac{t^\nu}{\nu!}.$$

A special case of such a sum we shall consider in the next paragraph. In the sequel we shall sum also some related series.

II

Let

$$F(x)=[x(x-1)\dots(x-k)]^{-n}\qquad(k,n\geqslant 1). \tag{6}$$

If $n=1$ we have (see [1], p. 392)

$$\sum_{x=k+1}^{\infty}F(x)=\sum_{\nu=1}^{\infty}\frac{1}{\nu(\nu+1)\dots(\nu+k)}=\frac{1}{k!\,k}. \tag{7}$$

If $n=2$ we have (see [1], p. 394)

$$\sum_{x=k+1}^{\infty}F(x)=\sum_{\nu=1}^{\infty}\frac{1}{[\nu(\nu+1)\dots(\nu+k)]^2} \tag{8}$$

$$=\frac{(2k-1)!!\,2^k}{(k!)^3}\left[\frac{\pi^2}{6}-3\sum_{\nu=1}^{k}\frac{1}{\nu\cdot 2^\nu}\,\frac{(\nu-1)!}{(2\nu-1)!!}\right].$$

If $k=1$ we have (see [2], [3])

$$\sum_{\nu=1}^{\infty}\frac{1}{\nu^n(\nu+1)^n}=(-1)^n\sum_{s=1}^{[n/2]}\frac{(2n-2s-1)!\,(2\pi)^{2s}}{(n-1)!\,(n-2s)!\,(2s)!}\,|B_{2s}|+(-1)^{n-1}\binom{2n-1}{n-1}. \tag{9}$$

We consider the case $n(k-1)\equiv 0 \pmod 2$. The decomposition of $F(x)$ into partial fractions is

$$F(x)=\sum_{i=0}^{n-1}\left[\frac{A_i^0}{x^{n-i}}+\frac{A_i^1}{(x-1)^{n-i}}+\dots+\frac{A_i^k}{(x-k)^{n-i}}\right]. \tag{10}$$

The coefficients A_i^j depend also on n and k but we go on with simplified notations. We shall prove that

$$A_j^i=(-1)^{nk+i}A_i^{k-j}. \tag{11}$$

Let

$$F_j(x)\equiv(x-j)^n F(x)=\sum_{i=0}^{\infty}a_i^j(x-j)^i, \tag{12}$$

$$F_{k-j}(x)=[x-(k-j)]^n F(x)=\sum_{i=0}^{\infty}a_i^{k-j}[x-(k-j)]^i. \tag{13}$$

From (10), (12) and (13) we get

$$a_i^j = A_i^j \qquad (i=0,\,1,\,\ldots,\,n-1), \tag{14}$$

$$a_i^{k-j} = A_i^{k-j} \qquad (i=0,\,1,\,\ldots,\,n-1). \tag{15}$$

Since $F_j(x) = (-1)^{nk} F_{k-j}(k-x)$ we find that $a_i^j = (-1)^{nk+i} a_i^{k-j}$. Using (14) and (15) we establish the validity of (11). By assumption $n(k-1) \equiv 0 \pmod 2$, so that instead of (11) we can write

$$A_i^j = (-1)^{n-i} A_i^{k-j}. \tag{16}$$

Summing over x from $k+1$ to ∞ we get

$$\begin{aligned}\sum_{x=k+1}^{\infty} F(x) &= \sum_{\nu=1}^{\infty} [\nu(\nu+1)\ldots(\nu+k)]^{-n}\\ &= \sum_{i=0}^{n-2} [A_i^0 \zeta(n-i) + A_i^1 \zeta(n-i) + \cdots + A_i^k \zeta(n-i)]\\ &\quad - \sum_{i=0}^{n-2} \left[A_i^0 \sum_{x=1}^{k} \frac{1}{x^{n-i}} + A_i^1 \sum_{x=2}^{k} \frac{1}{(x-1)^{n-i}} + \cdots + A_i^{k-1} \sum_{x=k}^{k} \frac{1}{(x-k+1)^{n-i}}\right]\\ &\quad + \sum_{x=k+1}^{\infty} \left[\frac{A_{n-1}^0}{x} + \frac{A_{n-1}^1}{x-1} + \cdots + \frac{A_{n-1}^k}{x-k}\right].\end{aligned}$$

But $A_{n-1}^0 + A_{n-1}^1 + \cdots + A_{n-1}^k = 0$ so that

$$\begin{aligned}\sum_{\nu=1}^{\infty} [\nu(\nu+1)\ldots(\nu+k)]^{-n} &= \sum_{i=0}^{n-2} (A_i^0 + A_i^1 + \cdots + A_i^k)\, \zeta(n-i)\\ &\quad - \sum_{i=0}^{n-2} \left[A_i^0 \sum_{x=1}^{k} \frac{1}{x^{n-i}} + A_i^1 \sum_{x=1}^{k-1} \frac{1}{x^{n-i}} + \cdots + A_i^{k-1} \sum_{x=1}^{1} \frac{1}{x^{n-i}}\right]\\ &\quad - \left[A_{n-1}^0 \sum_{x=1}^{k} \frac{1}{x} + A_{n-1}^1 \sum_{x=1}^{k-1} \frac{1}{x} + \cdots + A_{n-1}^{k-1} \sum_{x=1}^{1} \frac{1}{x}\right].\end{aligned}$$

If $n-i$ is odd then $A_i^0 + A_i^1 + \cdots + A_i^k = 0$ by (16). Hence, we have

$$\begin{aligned}\sum_{\nu=1}^{\infty} [\nu(\nu+1)\ldots(\nu+k)]^{-n} &= \sum_{i=1}^{[n/2]} (A_{n-2i}^0 + A_{n-2i}^1 + \cdots + A_{n-2i}^k)\, \zeta(2i)\\ &\quad - \sum_{i=0}^{n-1} \left[A_i^0 \sum_{x=1}^{k} \frac{1}{x^{n-i}} + A_i^1 \sum_{x=1}^{k-1} \frac{1}{x^{n-i}} + \cdots + A_i^{k-1} \sum_{x=1}^{1} \frac{1}{x^{n-i}}\right].\end{aligned}$$

Making use of (5) and introducing the following notations

$$\textstyle\sum_m^i = \displaystyle\sum_{x=1}^{m} \frac{1}{x^i} \tag{17}$$

we can write the last equality in the form

$$\begin{aligned}\sum_{\nu=1}^{\infty} [\nu(\nu+1)\ldots(\nu+k)]^{-n} &= \sum_{i=1}^{[n/2]} (A_{n-2i}^0 + A_{n-2i}^1 + \cdots + A_{n-2i}^k)\, \frac{2^{2i-1}}{(2i)!}\, \pi^{2i}\, |B_{2i}|\\ &\quad - \sum_{i=0}^{n-1} (A_i^0 \textstyle\sum_k^{n-i} + A_i^1 \sum_{k-1}^{n-i} + \cdots + A_i^{k-1} \sum_1^{n-i}).\end{aligned} \tag{18}$$

We remember that $n(k-1)\equiv 0 \pmod 2$ and that A_i^j $(j=0, 1, \ldots, n-1)$ are the first n coefficients in the expansion in powers of $(x-j)$ of $(x-j)^n F(x)$.

III

Now we take $n=2$ in (18) and we compare this result with (8). In this way we obtain an interesting identity.

We have from (18)

$$(19) \qquad \sum_{\nu=1}^{\infty} \frac{1}{[\nu(\nu+1)\ldots(\nu+k)]^2} = (A_0^0 + A_0^1 + \cdots + A_0^k)\,\frac{\pi^2}{6}$$
$$-(A_0^0 \textstyle\sum_k^2 + A_0^1 \sum_{k-1}^2 + \cdots + A_0^{k-1} \sum_1^2)$$
$$-(A_1^0 \textstyle\sum_k^1 + A_1^1 \sum_{k-1}^1 + \cdots + A_1^{k-1} \sum_1^1).$$

The coefficients A_0^j and A_1^j we determine from

$$\frac{1}{[x(x-1)\ldots(x-j+1)(x-j-1)\ldots(x-k)]^2} = A_0^j + A_1^j(x-j) + \cdots$$

i.e.

$$x^2(x-1)^2\ldots(x-j+1)^2(x-j-1)^2\ldots(x-k)^2[A_0^j + A_1^j(x-j) + \cdots] = 1,$$

or equivalently

$$(t+j)^2(t+j-1)^2\ldots(t+1)^2(t-1)^2\ldots(t-k+j)^2(A_0^j + A_1^j t + \cdots) = 1.$$

It follows that

$$(j!)^2[(k-j)!]^2 A_0^j = 1,$$

$$(j!)^2[(k-j)!]^2\left[A_1^j + 2\left(\frac{1}{j} + \frac{1}{j-1} + \cdots + 1 - 1 - \frac{1}{2} - \cdots - \frac{1}{k-j}\right)A_0^j\right] = 0.$$

Hence

$$(20) \qquad A_0^j = \frac{1}{(j!)^2[(k-j)!]^2} = \frac{1}{(k!)^2}\binom{k}{j}^2,$$

$$(21) \qquad A_1^j = 2A_0^j\left(\textstyle\sum_{k-j}^1 - \sum_j^1\right) = \frac{2}{(k!)^2}\binom{k}{j}^2\left(\textstyle\sum_{k-j}^1 - \sum_j^1\right).$$

From (20) we obtain

$$(22) \qquad A_0^1 + A_0^2 + \cdots + A_0^k = \frac{1}{(k!)^2}\sum_{j=0}^{k}\binom{k}{j}^2 = \frac{1}{(k!)^2}\binom{2k}{k}$$
$$= \frac{(2k)!}{(k!)^4} = \frac{2^k(2k-1)!!}{(k!)^3}.$$

Substituting (20), (21) and (22) into (19) we get

$$(23) \qquad \sum_{\nu=1}^{\infty}\frac{1}{[\nu(\nu+1)\ldots(\nu+k)]^2} = \frac{2^k\cdot(2k-1)!!}{(k!)^3}\,\frac{\pi^2}{6} - \frac{1}{(k!)^2}\sum_{j=1}^{k}\binom{k}{j}^2 \textstyle\sum_j^2$$
$$-\frac{2}{(k!)^2}\sum_{j=1}^{k}\binom{k}{j}^2\left(\textstyle\sum_j^1 - \sum_{k-j}^1\right)\sum_j^1.$$

The comparison of (8) and (23) furnishes the following identity

$$\sum_{j=1}^{k}\binom{k}{j}^2\sum\nolimits_j^2+2\sum_{j=1}^{k}\binom{k}{j}^2\left(\sum\nolimits_j^1-\sum\nolimits_{k-j}^1\right)\sum\nolimits_j^1=3\binom{2k}{k}\sum_{j=1}^{k}\frac{2}{j\,2^j}\,\frac{(j-1)\,!}{(2j-1)\,!!}\tag{24}$$

where

$$\sum\nolimits_j^1=1+\frac{1}{2}+\cdots+\frac{1}{j},\qquad \sum\nolimits_j^2=1+\frac{1}{2^2}+\cdots+\frac{1}{j^2}.$$

IV

We shall give an example. Let $n=3$, $k=$odd. Then (18) gives

$$\begin{aligned}\sum_{\nu=1}^{\infty}\frac{1}{[\nu(\nu+1)\ldots(\nu+k)]^3}&=(A_1^0+A_1^1+\cdots+A_1^k)\frac{\pi^2}{6}\\&-(A_0^0\textstyle\sum_k^3+A_0^1\sum_{k-1}^3+\cdots+A_0^{k-1}\sum_1^3)\\&-(A_1^0\textstyle\sum_k^2+A_1^1\sum_{k-1}^2+\cdots+A_1^{k-1}\sum_1^2)\\&-(A_2^0\textstyle\sum_k^1+A_2^1\sum_{k-1}^1+\cdots+A_2^{k-1}\sum_1^1).\end{aligned}\tag{25}$$

If $k=1$ (25) reduces to a special case of formula (9).
If $k=3$ we have

$$\begin{aligned}\sum_{\nu=1}^{\infty}\frac{1}{[\nu(\nu+1)(\nu+2)(\nu+3)]^3}&=(A_1^0+A_1^1+A_1^2+A_1^3)\frac{\pi^2}{6}\\&-\left[A_0^0\left(1+\frac{1}{8}+\frac{1}{27}\right)+A_0^1\left(1+\frac{1}{8}\right)+A_0^2\right]\\&-\left[A_1^0\left(1+\frac{1}{4}+\frac{1}{9}\right)+A_1^1\left(1+\frac{1}{4}\right)+A_1^2\right]\\&-\left[A_2^0\left(1+\frac{1}{2}+\frac{1}{3}\right)+A_2^1\left(1+\frac{1}{2}\right)+A_2^2\right].\end{aligned}\tag{26}$$

The coefficients A_i^0 and A_i^1 $(i=0, 1, 2)$ we determine from identities

$$(x-1)^3(x-2)^3(x-3)^3(A_0^0+A_1^0x+A_2^0x^2+\cdots)=1,$$

$$(x+1)^3(x-1)^3(x-2)^3(A_0^1+A_1^1x+A_2^1x^2+\cdots)=1.$$

We find in this way

$$\begin{aligned}&A_0^0=-\frac{1}{216},\qquad A_1^0=-\frac{11}{432},\qquad A_2^0=-\frac{103}{1296},\\&A_0^1=\frac{1}{8},\qquad A_1^1=\frac{3}{16},\qquad A_2^1=\frac{9}{16}.\end{aligned}\tag{27}$$

Making use of (16) we determine the remaining coefficients

$$\begin{aligned}&A_0^2=-\frac{1}{8},\qquad A_1^2=\frac{3}{16},\qquad A_2^2=-\frac{9}{16},\\&A_0^3=\frac{1}{216},\qquad A_1^3=-\frac{11}{432},\qquad A_2^3=\frac{103}{1296}.\end{aligned}\tag{28}$$

Putting the values (27) and (28) into (26) we obtain

$$\sum_{\nu=1}^{\infty} \frac{1}{[\nu(\nu+1)(\nu+2)(\nu+3)]^3} = \frac{35}{648}\pi^2 - \frac{6217}{11664}. \tag{29}$$

The series on the left-hand side of (29) converges very rapidly. If we take only the first term of this series into account, we get

$$\pi^2 \approx \frac{198971}{20160} = 9{,}869593\ldots \tag{30}$$

which is very close to exact value

$$\pi^2 = 9{,}869604\ldots \tag{31}$$

In the case when k is an arbitrary odd positive integer the coefficients A_i^j in formula (25) can be expressed by means of Stirling numbres of the first kind. Indeed, we have

$$[(t+j)(t+j-1)\ldots(t+1)(t-1)(t-2)\ldots \tag{32}$$
$$(t-k+j)]^3 (A_0^j + A_1^j t + A_2^j t^2 + \cdots) = 1.$$

By definition of Stirling numbers S_p^ν:

$$x(x-1)\ldots(x-p+1) = \sum_{\nu=1}^{p} S_p^\nu x^\nu.$$

Using this, (32) becomes

$$\left[\sum_{\nu=1}^{j+1} S_{j+1}^\nu (-t)^{\nu-1}\right]^3 \left(\sum_{\nu=1}^{k-j+1} S_{k-j+1}^\nu t^{\nu-1}\right)^3 (A_0^j + A_1^j t + A_2^j t^2 + \cdots) = (-1)^j. \tag{33}$$

Since $(a+bt+ct^2+\cdots)^3 = a^3 + 3a^2bt + 3a(b^2+ac)t^2 + \cdots$ (33) gives

$$[S_{j+1}^{1\,3} - 3S_{j+1}^{1\,2} S_{j+1}^2 t + 3S_{j+1}^1 (S_{j+1}^{2\,2} + S_{j+1}^1 S_{j+1}^3) t^2 + \cdots]$$
$$\times [S_{k-j+1}^{1\,3} + 3S_{k-j+1}^{1\,2} S_{k-j+1}^2 t + 3S_{k-j+1}^1 (S_{k-j+1}^{2\,2} + S_{k-j+1}^1 S_{k-j+1}^3) t^2 + \cdots]$$
$$\times (A_0^j + A_1^j t + A_2^j t^2 + \cdots) = (-1)^j$$

where $S_p^{\nu\,m} = (S_p^\nu)^m$. Multiplying two square brackets, we get

$$\{S_{j+1}^{1\,3} S_{k-j+1}^{1\,3} + 3S_{j+1}^{1\,2} S_{k-j+1}^{1\,2} (S_{j+1}^1 S_{k-j+1}^2 - S_{j+1}^2 S_{k-j+1}^1) t +$$
$$+ 3S_{j+1}^1 S_{k-j+1}^1 [S_{j+1}^{1\,2} (S_{k-j+1}^{2\,2} + S_{k-j+1}^1 S_{k-j+1}^3) + S_{k-j+1}^{1\,2} (S_{j+1}^{2\,2} + S_{j+1}^1 S_{j+1}^3) -$$
$$- 3S_{j+1}^1 S_{j+1}^2 S_{k-j+1}^1 S_{k-j+1}^2] t^2 + \cdots\} \times (A_0^j + A_1^j t + A_2^j t^2 + \cdots) = (-1)^j.$$

Comparing the coefficients of powers of t, we find

$$S_{j+1}^{1\,3} S_{k-j+1}^{1\,3} A_0^j = (-1)^j,$$
$$S_{j+1}^{1\,3} S_{k-j+1}^{1\,3} A_1^j + 3S_{j+1}^{1\,2} S_{k-j+1}^{1\,2} (S_{j+1}^1 S_{k-j+1}^2 - S_{j+1}^2 S_{k-j+1}^1) A_0^j = 0,$$
$$S_{j+1}^{1\,3} S_{k-j+1}^{1\,3} A_2^j + 3S_{j+1}^{1\,2} S_{k-j+1}^{1\,2} (S_{j+1}^1 S_{k-j+1}^2 - S_{j+1}^2 S_{k-j+1}^1) A_1^j +$$
$$+ 3S_{j+1}^1 S_{k-j+1}^1 [S_{j+1}^{1\,2} (S_{k-j+1}^{2\,2} S_{k-j+1}^1 + S_{k-j+1}^3) + S_{k-j+1}^{1\,2} (S_{j+1}^{2\,2} + S_{j+1}^1 S_{j+1}^3) -$$
$$- 3S_{j+1}^1 S_{j+1}^2 S_{k-j+1}^1 S_{k-j+1}^2] A_0^j = 0.$$

Finally, from this system we obtain

$$
\begin{aligned}
&A_0^j=(-1)^j/(S_{j+1}^1\, S_{k-j+1}^1)^3,\\
(34)\qquad &A_1^j=(-1)^j\, 3\,(S_{j+1}^2\, S_{k-j+1}^1-S_{j+1}^1\, S_{k-j+1}^2)/(S_{j+1}^1\, S_{k-j+1}^1)^4,\\
&A_2^j=(-1)^j\, 3\, M_j/(S_{j+1}^1\, S_{k-j+1}^1)^5,
\end{aligned}
$$

where

$$
\begin{aligned}
M_j=2\,(S_{j+1}^1\, S_{k-j+1}^2-S_{j+1}^2\, S_{k-j+1}^1)^2-S_{j+1}^1\, S_{k-j+1}^1\,(S_{j+1}^1\, S_{k-j+1}^3&\\
+S_{j+1}^3\, S_{k-j+1}^1-S_{j+1}^2\, S_{k-j+1}^2)&\,.
\end{aligned}
$$

We notify that $S_p^1=(-1)^{p-1}(p-1)!$

V

Let n and k be even. We shall write $2n$ and $2k$ instead of n and k respectively:

$$
\begin{aligned}
(35)\qquad F(x)&=[x\,(x-1)\;\ldots\;(x-2k)]^{-2n}\\
&=\sum_{i=0}^{2n-1}\left[\frac{A_i^0}{x^{2n-i}}+\frac{A_i^1}{(x-1)^{2n-i}}+\cdots+\frac{A_i^{2k}}{(x-2k)^{2n-i}}\right].
\end{aligned}
$$

Instead of (16) we have

$$
(36)\qquad A_i^j=(-1)^i A_i^{2k-j}.
$$

Multiplying both sides of (35) by $(-1)^{x-1}$ and summing over x from $2k+1$ to ∞ we get

$$
\begin{aligned}
\sum_{x=2k+1}^{\infty}(-1)^{x-1}F(x)&=\sum_{\nu=1}^{\infty}(-1)^{\nu-1}[\nu\,(\nu+1)\;\ldots\;(\nu+2k)]^{-2n}\\
&=\sum_{i=0}^{2n-2}(A_i^0-A_i^1+A_i^2-\cdots+A_i^{2k})\,(1-2^{1-2n+i})\,\zeta\,(2n-i)\\
-\sum_{i=0}^{2n-1}&\left[A_i^0\sum_{x=1}^{2k}\frac{(-1)^{x-1}}{x^{2n-i}}-A_i^1\sum_{x=2}^{2k}\frac{(-1)^{x-2}}{(x-1)^{2n-i}}+\cdots-A_i^{2k-1}\sum_{x=2k}^{2k}\frac{(-1)^{x-2k}}{(x-2k+1)^{2n-i}}\right]
\end{aligned}
$$

since

$$
(37)\qquad \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{\nu^s}=(1-2^{1-s})\,\zeta\,(s)\,.
$$

If i is odd then $A_i^0-A_i^1+A_i^2-\cdots+A_i^{2k}=0$ by (36) so that

$$
\begin{aligned}
\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{[\nu\,(\nu+1)\;\ldots\;(\nu+2k)]^{2n}}&=\sum_{i=1}^{n}(A_{2n-2i}^0-A_{2n-2i}^1+\cdots+A_{2n-2i}^{2k})\,(1-2^{1-2i})\,\zeta\,(2i)\\
-\sum_{i=0}^{2n-1}&\left[A_i^0\sum_{x=1}^{2k}\frac{(-1)^{x-1}}{x^{2n-i}}-A_i^1\sum_{x=1}^{2k-1}\frac{(-1)^{x-1}}{x^{2n-i}}+\cdots-A_i^{2k-1}\sum_{x=1}^{1}\frac{(-1)^{x-1}}{x^{2n-i}}\right].
\end{aligned}
$$

Using (5) and putting

$$
(38)\qquad \sigma_m^i=1-\frac{1}{2^i}+\frac{1}{3^i}-\cdots+(-1)^{m-1}\frac{1}{m^i}\,,
$$

we obtain

$$
\begin{aligned}
(39)\qquad \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{[\nu\,(\nu+1)\;\ldots\;(\nu+2k)]^{2n}}&=\sum_{i=1}^{n}(A_{2n-2i}^0-A_{2n-2i}^1+\cdots+\\
+A_{2n-2i}^{2k})\,\frac{2^{2i-1}-1}{(2i)!}\,\pi^{2i}\,|\,B_{2i}\,|&-\sum_{i=0}^{2n-1}(A_i^0\,\sigma_{2k}^{2n-i}-A_i^1\,\sigma_{2k-1}^{2n-i}+\cdots-A_i^{2k-1}\,\sigma_1^{2n-i}).
\end{aligned}
$$

For instance, if $n=1$ we get

$$\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{[\nu(\nu+1)\cdots(\nu+2k)]^2}=(A_0^0-A_0^1+A_0^2-\cdots+A_0^{2k})\frac{\pi^2}{12} \tag{40}$$
$$-(A_0^0\sigma_{2k}^2-A_0^1\sigma_{2k-1}^2+\cdots-A_0^{2k-1}\sigma_1^2)$$
$$-(A_1^0\sigma_{2k}^1-A_1^1\sigma_{2k-1}^1+\cdots-A_1^{2k-1}\sigma_1^1).$$

The coefficients A_0^j and A_1^j we can find from (20) and (21) putting $2k$ instead of k. Hence,

$$A_0^j=\frac{1}{[(2k)!]^2}\binom{2k}{j}^2, \tag{41}$$

$$A_1^j=\frac{2}{[(2k)!]^2}\binom{2k}{j}^2(\Sigma_{2k-1}^1-\Sigma_j^1). \tag{42}$$

From the well known identity

$$\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}^2=(-1)^k\binom{2k}{k}$$

we conclude that

$$A_0^0-A_0^1+A_0^2-\cdots+A_0^{2k}=\frac{(-1)^k}{[(2k)!]^2}\binom{2k}{k}. \tag{43}$$

Putting (41), (42) and (43) into (40) we get

$$\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{[\nu(\nu+1)\cdots(\nu+2k)]^2}=\frac{(-1)^k}{[(2k)!]^2}\binom{2k}{k}\frac{\pi^2}{12} \tag{44}$$
$$-\frac{1}{[(2k)!]^2}\sum_{j=0}^{2k-1}(-1)^j\binom{2k}{j}^2\sigma_{2k-j}^2$$
$$-\frac{2}{[(2k)!]^2}\sum_{j=0}^{2k-1}(-1)^j\binom{2k}{j}^2(\Sigma_{2k-j}^1-\Sigma_j^1)\sigma_{2k-j}^1.$$

Thus for $k=1$ we obtain

$$\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{\nu^2(\nu+1)^2(\nu+2)^2}=\frac{7}{16}-\frac{\pi^2}{24}. \tag{45}$$

VI

A new type of series we shall consider in this paragraph. We assume that $n(k-1)\equiv 0 \pmod 2$. Changing x to $x-\frac{1}{2}$ in (10) we get

$$F\left(x-\frac{1}{2}\right)=\sum_{i=0}^{n-1}\left[\frac{A_i^0}{\left(x-\frac{1}{2}\right)^{n-i}}+\frac{A_i^1}{\left(x-\frac{3}{2}\right)^{n-i}}+\cdots+\frac{A_i^k}{\left(x-\frac{2k+1}{2}\right)^{n-i}}\right]. \tag{46}$$

Summing over x from $k+1$ to ∞ and taking into account that

$$\sum_{x=1}^{\infty}\frac{1}{\left(x-\frac{1}{2}\right)^s}=2^s\sum_{x=1}^{\infty}\frac{1}{(2x-1)^s}=2^s(1-2^{-s})\zeta(s)=(2^s-1)\zeta(s) \tag{47}$$

we obtain

$$\sum_{x=k+1}^{\infty} F\left(x-\frac{1}{2}\right)=2^{n(k+1)} \sum_{\nu=0}^{\infty} \frac{1}{[(2\nu+1)(2\nu+3)\cdots(2\nu+2k+1)]^n}$$

$$=\sum_{i=0}^{n-2} (A^0+A_i^1+\cdots+A_i^k)(2^{n-i}-1)\zeta(n-i)$$

$$-\sum_{i=0}^{n-1}\left[A_i^0 \sum_{x=1}^{k} \frac{1}{\left(x-\frac{1}{2}\right)^{n-i}}\right.$$

$$\left.+A_i^1 \sum_{x=2}^{k} \frac{1}{\left(x-\frac{3}{2}\right)^{n-i}}+\cdots+A_i^{k-1}\sum_{x=k}^{k} \frac{1}{\left(x-\frac{2k-1}{2}\right)^{n-i}}\right].$$

Using (16) we get

$$2^{n(k+1)} \sum_{\nu=0}^{\infty} \frac{1}{[(2\nu+1)(2\nu+3)\cdots(2\nu+2k+1)]^n}$$

$$=\sum_{i=1}^{[n/2]} (A_{n-2i}^0+A_{n-2i}^1+\cdots+A_{n-2i}^k)\frac{2^{2i}-1}{(2i)!}\, 2^{2i-1}\pi^{2i}|B_{2i}|$$

$$-\sum_{i=0}^{n-1} 2^{n-i}\left[A_i^0 \sum_{x=1}^{k} \frac{1}{(2x-1)^{n-i}}\right.$$

$$\left.+A_i^1 \sum_{x=1}^{k-1} \frac{1}{(2x-1)^{n-i}}+\cdots+A_i^{k-1}\sum_{x=1}^{1} \frac{1}{(2x-1)^{n-i}}\right].$$

Finally, with the notations

$$\Theta_m^i=1+\frac{1}{3^i}+\cdots+\frac{1}{(2m-1)^i} \tag{48}$$

the last formula becomes

$$2^{n(k+1)} \sum_{\nu=0}^{\infty} \frac{1}{[(2\nu+1)(2\nu+3)\cdots(2\nu+2k+1)]^n} \tag{49}$$

$$=\sum_{i=1}^{[n/2]} (A_{n-2i}^0+A_{n-2i}^1+\cdots+A_{n-2i}^k)\frac{2^{2i-1}(2^{2i}-1)}{(2i)!}\pi^{2i}|B_{2i}|$$

$$-\sum_{i=0}^{n-1} 2^{n-i}(A_i^0\Theta_k^{n-i}+A_i^1\Theta_{k-1}^{n-i}+\cdots+A_i^{k-1}\Theta_1^{n-i})$$

where $n(k-1)\equiv 0 \pmod 2$.

If $n=2$ we find

$$2^{2k+2} \sum_{\nu=0}^{\infty} \frac{1}{[(2\nu+1)(2\nu+3)\ \ldots\ (2\nu+2k+1)]^2}$$

$$=(A_0^0+A_0^1+\cdots+A_0^k)\frac{\pi^2}{2}$$

$$-4(A_0^0\Theta_k^2+A_0^1\Theta_{k-1}^2+\cdots+A_0^{k-1}\Theta_1^2)$$

$$-2(A_1^0\Theta_k^1+A_1^1\Theta_{k-1}^1+\cdots+A_1^{k-1}\Theta_1^1).$$

We can substitute A_i^j from (20) and (21) in the last formula. Hence

$$2^{2k+2}\sum_{\nu=0}^{\infty}\frac{1}{[(2\nu+1)(2\nu+3)\ \dots\ (2\nu+2k+1)]^2} \tag{50}$$
$$=\frac{1}{(k!)^2}\binom{2k}{k}\frac{\pi^2}{2}-\frac{4}{(k!)^2}\sum_{j=0}^{k-1}\binom{k}{j}^2\Theta_{k-j}^2$$
$$-\frac{4}{(k!)^2}\sum_{j=0}^{k-1}\binom{k}{j}^2\left(\sum_{k-j}^1-\sum_j^1\right)\Theta_{k-j}^1\,.$$

For $k=1$ we obtain a known result (see [3])

$$\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+1)^2(2\nu+3)^2}=\sum_{\nu=1}^{\infty}\frac{1}{(4\nu^2-1)^2}=\frac{\pi^2}{16}-\frac{1}{2}\,. \tag{51}$$

For $k=2$ we get (see [5], p. 271)

$$\sum_{\nu=0}^{\infty}\frac{1}{[(2\nu+1)(2\nu+3)(2\nu+5)]^2}=\frac{3\pi^2}{256}-\frac{1}{9}\,. \tag{52}$$

VII

We shall consider yet another type of series. We suppose that $(n+1)(k+1)$ is even i.e. at least one of numbers n, k is odd. We start from identity (46). Multiplying this identity by $(-1)^{x-1}$ and summing over x from $k+1$ to ∞ we get

$$\sum_{x=k+1}^{\infty}(-1)^{x-1}F\left(x-\frac{1}{2}\right)=\sum_{i=0}^{n-1}[A_i^0-A_i^1+A_i^2-\cdots+(-1)^kA_i^k]\,2^{n-i}\eta(n-i) \tag{53}$$
$$-\sum_{i=0}^{n-1}2^{n-i}\left[A_i^0\sum_{x=1}^{k}\frac{(-1)^{x-1}}{(2x-1)^{n-i}}-A_i^1\sum_{x=1}^{k-1}\frac{(-1)^{x-1}}{(2x-1)^{n-i}}\right.$$
$$\left.+\cdots+(-1)^{k-1}A_i^{k-1}\sum_{x=1}^{1}\frac{(-1)^{x-1}}{(2x-1)^{n-i}}\right],$$

where

$$\eta(s)=1-\frac{1}{3^s}+\frac{1}{5^s}-\cdots \tag{54}$$

Using (11) and the fact that $(n+1)(k+1)$ is even and assuming that $n-i$ is even, we obtain

$$(-1)^jA_i^j+(-1)^{k-j}A_i^{k-j}=(-1)^jA_i^{k-j}[(-1)^{nk+i}+(-1)^k]$$
$$=(-1)^{k+j}A_i^{k-j}[1+(-1)^{nk+k+i}]$$
$$=(-1)^{k+j}A_i^{k-j}[1+(-1)^{nk+k+n}]$$
$$=(-1)^{k+j}A_i^{k-j}[1+(-1)^{(n+1)(k+1)-1}]$$
$$=0$$

so that $A_i^0 - A_i^1 + \cdots + (-1)^k A_i^k = 0$ for $n-i$ even. Hence, we have

$$\sum_{x=k+1}^{\infty} (-1)^{x-1} F\left(x - \frac{1}{2}\right) = (-1)^k 2^{n(k+1)} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{[(2\nu+1)(2\nu+3)\cdots(2\nu+2k+1)]^n}$$

$$= \sum_{i=0}^{\left[\frac{n-1}{2}\right]} [A_{n-2i-1}^0 - A_{n-2i-1}^1 + \cdots + (-1)^k A_{n-2i-1}^k] \, 2^{2i+1} \eta(2i+1)$$

$$- \sum_{\nu=1}^{n-1} 2^{n-i} [A_i^0 \theta_k^{n-i} - A_i^1 \theta_{k-1}^{n-i} + \cdots + (-1)^{k-1} A_i^{k-1} \theta_1^{n-i}]$$

with

(55) $$\theta_m^i = 1 - \frac{1}{3^i} + \frac{1}{5^i} - \cdots + (-1)^{m-1} \frac{1}{(2m-1)^i}.$$

Since (see [4])

$$\eta(2s+1) = \frac{\pi^{2s+1}}{2^{2s+2}(2s)!} |E_{2s}|$$

where E_s are Eulerian numbers defined by

$$(\operatorname{ch} t)^{-1} = \sum_{\nu=0}^{\infty} E_\nu t^\nu/\nu!,$$

we can write

(56) $$(-1)^k 2^{n(k+1)} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{[(2\nu+1)(2\nu+3)\cdots(2\nu+2k+1)]^n}$$

$$= \sum_{i=0}^{\left[\frac{n-1}{2}\right]} [A_{n-2i-1}^0 - A_{n-2i-1}^1 + \cdots + (-1)^k A_{n-2i-1}^k] \frac{\pi^{2i+1}}{2 \cdot (2i)!} |E_{2i}|$$

$$- \sum_{i=0}^{n-1} 2^{n-i} [A_i^0 \theta_k^{n-i} - A_i^1 \theta_{k-1}^{n-i} + \cdots + (-1)^{k-1} A_i^{k-1} \theta_1^{n-i}].$$

We remember that in the last formula n or k must be odd.

If $n=1$ we obtain from (56)

$$(-1)^k 2^{k+1} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)(2\nu+3)\cdots(2\nu+2k+1)} = [A_0^0 - A_0^1 + \cdots + (-1)^k A_0^k] \frac{\pi}{2}$$

$$-2[A_0^0 \theta_k^1 - A_0^1 \theta_{k-1}^1 + \cdots + (-1)^{k-1} A_0^{k-1} \theta_1^1].$$

From identity

$$x(x-1)\cdots(x-j+1)(x-j-1)\cdots(x-k)[A_0^j + A_1^j(x-j) + \cdots] = 1$$

we find

$$A_0^j = \frac{(-1)^{k-j}}{k!} \binom{k}{j}; \quad \sum_{j=0}^{k} (-1)^j A_0^j = (-1)^k \frac{2^k}{k!}.$$

Using this our formula becomes

(57) $$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)(2\nu+3)\cdots(2\nu+2k+1)} = \frac{1}{k!}\left[\frac{\pi}{4} - \frac{1}{2^k} \sum_{j=1}^{k} \binom{k}{j} \theta_j^1\right].$$

This formula holds for $k=0, 1, 2, \ldots$ For $k=0$ and $k=1$ we obtain respectively

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{2\nu+1}=\frac{\pi}{4}, \quad \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{4\nu^2-1}=\frac{\pi}{4}-\frac{1}{2}. \tag{58}$$

If $n=2$, k must be odd. From (56) we obtain that in this case

$$(-1)^k\, 2^{2k+2} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{[(2\nu+1)(2\nu+3)\cdots(2\nu+2k+1)]^2} \tag{59}$$

$$=(A_1^0-A_1^1+\cdots-A_1^k)\frac{\pi}{2}-4\sum_{j=0}^{k-1}(-1)^j A_0^j\, \theta_{k-j}^2-2\sum_{j=1}^{k-1}(-1)^j A_1^j\, \theta_{k-j}^1 .$$

Using (20) and (21) formula (59) becomes

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{[(2\nu+1)(2\nu+3)\cdots(2\nu+2k+1)]^2} \tag{60}$$

$$=\frac{\pi}{2^{2k+2}(k!)^2}\sum_{j=0}^{k}(-1)^j\binom{k}{j}^2\left(\sum_j^1-\sum_{k-j}^1\right)$$

$$-\frac{1}{2^{2k}(k!)^2}\sum_{j=1}^{k}(-1)^j\binom{k}{j}^2\left[\theta_j^2+\left(\sum_j^1-\sum_{k-j}^1\right)\theta_j^1\right],$$

where k is odd and $\sum_i^j$ and θ_i^j are defined by (17) and (55). For $k=1$ and $k=3$ we find from (60)

$$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{(4\nu^2-1)^2}=\frac{1}{2}-\frac{\pi}{8}, \tag{61}$$

$$\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{[(2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)]^2}=\frac{\pi}{1728}-\frac{7}{4050}. \tag{62}$$

If we put $k=1$ in general formula (56), we obtain the result of Kesava P. Menon [3]. Indeed, from

$$\frac{1}{x^n(x-1)^n}=\frac{1}{x^n}\sum_{\nu=0}^{n-1}(-1)^n\binom{n+\nu-1}{\nu}x^{\nu}+\frac{1}{(x-1)^n}\sum_{\nu=0}^{n-1}(-1)^{\nu}\binom{n+\nu-1}{\nu}(x-1)^{\nu}$$

we get

$$A_i^0=(-1)^n\binom{n+i-1}{i}, \quad A_i^1=(-1)^i\binom{n+i-1}{i}.$$

Formula (56) gives

$$-2^{2n}\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{(4\nu^2-1)^n}=(-1)^n\sum_{i=0}^{\left[\frac{n-1}{2}\right]}\binom{n+i-1}{i}\frac{\pi^{2i+1}}{(2i)!}|E_{2i}|$$

$$-(-1)^n\, 2^n\sum_{i=0}^{n-1}\frac{1}{2^i}\binom{n+i-1}{i}.$$

Finaly, using the identity (see [3])

$$\sum_{i=0}^{n-1}\frac{1}{2^i}\binom{n+i-1}{i}=2^{n-1}$$

we obtain

$$(63) \qquad \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{(4\nu^2-1)^n} = \frac{(-1)^{n-1}}{2^{2n}} \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n+i-1}{i} \frac{\pi^{2i+1}}{(2i)!} |E_{2i}| + \frac{(-1)^n}{2},$$

which is equivalent to Menon's result.

Remark 1. The series considered in this paper can be summed, in analogous way, without any restriction on parity of n and k, but in general case the sum is expressed in terms of ζ and η functions. Our formulas include all cases of those series in which the values of ζ and η functions can be expressed, in finite form, by π.

Remark 2. Consider the series $\sum_{\nu=0}^{\infty} (-1)^\nu/(2\nu+1)$ whose sum is $\pi/4$. The n-th partial sum of this series is denoted by θ_n^1. Since the series converges, it is also $(E, 1)$ — summable, i.e.

$$(64) \qquad \lim_{k\to\infty} U_k = \frac{\pi}{4}$$

where

$$U_k = \frac{1}{2^k} \sum_{j=1}^{k} \binom{k}{j} \theta_j^1 .$$

Using formula (57) we can express the difference $\frac{\pi}{4} - U_k$ in explicite form

$$(65) \qquad \frac{\pi}{4} - U_k = k! \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)(2\nu+3)\cdots(2\nu+2k+1)} .$$

Formula (64) follows immediately from (65). Moreover, formula (65) can be used to obtain an upper bound of $\frac{\pi}{4} - U_k$.

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