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ON THE FUNCTIONAL EQUATION: $T_1(t+s)\,T_2(t-s)=T_3(t)\,T_4(s)$

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In this paper we follow the terminology and symbolic of [2]. Thus, $R=\{t, s, \ldots\}$ denotes the set of all real numbers, $A=\{T, U, \ldots\}$ a Banach algebra with an identity I, $X=\{x, y, \ldots\}$ a Hilbert space with a scalar product (x, y) of two vectors $x, y \in X$. A function $T: R \rightarrow A$ is said to be regular if $T(t)$ is regular element of A for any $t \in R$.

The functional equation

$$T_1(t+s)\,T_2(t-s)=T_3(t)\,T_4(s) \tag{1}$$

was investigated by S. Kurepa in [2]. In this paper, among other results, we generalise to infinite dimensional spaces some results obtained in [2] for finite dimensional spaces.

We prove that functional equation (1) can be essentially reduced to the Cauchy functional equation and to the functional equation of the type

$$T(t+s)\,T(t-s)=V_1(\mathrm{t})\,V_2(s) \tag{2}$$

which was also solved in [2].

We have the following results:

Theorem 1. Let A be a Banach algebra with a unite I, R the set of all real numbers, $T_i: R \rightarrow A$ $(i=1, 2, 3, 4)$ a regular function such that

$$T_1(t+s)\,T_2(t-s)=T_3(t)\,T_4(s)$$

holds for all $t, s \in R$.

If $T_i(t)$ and $T_j(s)$ commute one with another for all $t, s \in R$ and $i, j=1, 2, 3, 4$, and if restrictions of T_1, T_2 and T_3 on an interval $\Delta=[a, b]$, $a<b$, are measurable in the Lebesgue sense, then these functions are of the following form:

$$\begin{aligned} T_1(t)&=T_1(0)\ \exp\ (t^2 T+t U_1),\\ T_2(t)&=T_2(0)\ \exp\ (t^2 T+t U_2),\\ T_3(t)&=T_3(0)\ \exp\ (2t^2 T+t U_3),\\ T_4(t)&=T_4(0)\ \exp\ (2t^2 T+t U_4),\\ U_3&=U_1+U_2,\quad U_4=U_1-U_2, \end{aligned} \tag{3}$$

where T, U_1 and U_2 are elements of A. They commute one with another as well as with $T_i(t)$ for $\mathrm{t} \in R$ and $i=1, 2, 3, 4$.

Theorem 2. Let X be a separable Hilbert space, A the Banach algebra of all continuous and linear operators from X in X, and $T_i : R \to A$ $(i = 1, 2, 3, 4)$ functions such that:

1. $$T_1(t+s)\, T_2(t-s) = T_3(t)\, T_4(s)$$

hold for all $t, s \in R$.

2. Zero is not in the spectrum of any $T_i(t)$ $(i = 1, 2, 3, 4;\ t \in R)$.

3. $(T_i(t)x, y)$ $(i = 1, 2, 3, 4)$ are measurable functions in the Lebesgue sense on R for all $x, y \in X$.

Under these conditions the functions $T_i(t)$ are strongly continuous on R.

Proof of the Theorem 1. Since functions

$$T_i(t)\, T_i^{-1}(0) \qquad (i = 1, 2, 3, 4)$$

satisfy all conditions of Theorem 1 we can without loss of generality assume that

$$T_i(0) = I \tag{4}$$

for $i = 1, 2, 3, 4$. Hence from now on we assume that (4) holds:

If we set $s = 0$ and $t = 0$ in (1) we get

$$T_1(t)\, T_2(t) = T_3(t), \quad T_1(s)\, T_2(-s) = T_4(s)$$

which together with (1) gives:

$$T_1(t+s)\, T_2(t-s) = T_1(t)\, T_2(t)\, T_1(s)\, T_2(-s). \tag{5}$$

If in (5) we replace s by $-s$ we get:

$$T_1(t-s)\, T_2(t+s) = T_1(t)\, T_2(t)\, T_1(-s)\, T_2(s)$$

which together with (5) leads to:

$$U(t+s)\, U(-s) = U(t-s)\, U(s), \quad U(0) = I, \tag{6}$$

where

$$U(t) = T_2(t)\, T_1^{-1}(t).$$

If we replace t by $t+s$ in (6) we get:

$$U(t+2s)\, U(-s) = U(t)\, U(s)$$

which together with (6) gives:

$$U(t+2s)\, U(t-s) = U(t+s)\, U(t). \tag{7}$$

Replacing in (7) s by $-s$ we get:

$$U(t-2s)\, U(t+s) = U(t-s)\, U(t)$$

which after multiplication with (7) implies

$$U(t+2s)\, U(t-2s) = U^2(t). \tag{8}$$

If we set $t = 2s$ in (8) we get:

$$U(2t) = U^2(t) \qquad (U(0) = I)$$

Denoting $u = t + 2s$, $v = t - 2s$ from (8) we get:

$$U(u)\, U(v) = U^2\left(\frac{u+v}{2}\right) = U(u+v)$$

i.e.

(9) $$U(t+s)=U(t)\,U(s)$$

for all $t, s\in R$.

Now, the measurability of T_1, T_2, T_3 on an interval implies the measurability of these functions on R ([2], Lemma 1). Furthermore, together with $t\to T_i(t)$, the functions $t\to T_i^{-1}(t)$ are measurable. Hence the function

$$t\to U(t)=T_2(t)\,T_1^{-1}(t)$$

is also measurable. This and (9) imply:

$$U(t)=\exp\ tT'$$

for any $t\in R$, where T' is an element of A ([1], pp. 280—291).

Thus

(10) $$T_2(t)=T_1(t)\ \exp\ tT'$$

If in (1) we replace $t+s$ by t, $t-s$ by s we get

(11) $$T_3(t+s)\,T_4(t-s)=T_1(2t)\,T_2(2s)$$

which in the same way leads to

(12) $$T_4(t)=T_3(t)\ \exp\ t\,T''$$

for any $t\in R$, where T'' is an element of A. Since T', T'' and $T_i(t)$ commute one with another, from (10) and (12) we deduce

$$T_1(t+s)\,T_1(t-s)=T_3(t)\,T_3(s)\ \exp\ [s(T'+T'')-t\,T]=$$
$$=T_3(t)\ \exp\ (-t\,T')\,T_3(s)\ \exp\ s(T'+T'')$$

i.e.

(13) $$T_1(t+s)\,T_1(t-s)=W_1(t)\,W_2(s)$$

where $W_1(t)=T_3(t)\ \exp\ (-t\,T')$ and $W_2(t)=T_3(t)\ \exp\ t(T'+T'')$. Since the functions T_1, W_1 and W_2 are measurable and $T_1(0)=W_1(0)=W_2(0)=I$, we have

$$T_1(t)=\exp\ (t^2T+t\,U_1),$$
$$W_1(t)=\exp\ (2\,t^2\,T+2\,t\,U_1)$$

([2], Theorem 2).

Thus (10) and (12) with a suitable notations imply (3). Q.E.D.

For the proof of Theorem 2 we need three lemmas.

L e m m a 1. Under conditions of Theorem 2 the functions

$$t\to T_i(t)\qquad (i=1, 2, 3, 4)$$

are locally bounded.

P r o o f: Due to the principal of uniform boundedness it is sufficient to prove that functions $t\to T_i(t)\ x$ are locally bounded for any $x\in X$. For given $a>0$ and $x\in X$, $x\neq 0$ we assert that $T_4(t)x$ is bounded on $[0, a]$. Otherwise one could find a sequence $s_k\in[0, a]$ such that

(14) $$\|T_4(s_k)\,x\|\to+\infty\quad \text{as}\quad k\to+\infty\,.$$

Take $b>16a$ and consider intervals $[0, b]$, $[a, b-a]$. Since functions

(15) $$t\to\|T_i(t)\|\quad (i=1, 2, 3, 4),\qquad t\to T_3(t)\,T_4(s_k)\,x,\quad k=1, 2, \ldots$$

are measurable on $[a, b-a]$ there exists a perfect set $F \subseteq [a, b-a]$ such that $mF > b-3a$ and function (15) are continuous on F. Thus there is a number $M > 0$ such that

$$\| T_i(t) \| \leqslant M \qquad (t \in F;\ i=1, 2, 3, 4).$$

Set:

$$E_k = \{ t \mid t \in F;\ \| T_3(t)\, T_4(s_j)\, x \| \geqslant (M^2+1) \| x \| \quad \text{for all } j \geqslant k \}.$$

Then:

a) $E_k \subseteq E_{k+1} \qquad (k=1, 2, \ldots)$

b) E_k is a closed set

c) $\| T_3(t)\, T_4(s_{k+i})\, x \| \geqslant (M^2+1) \| x \| \qquad (t \in E_k;\ i=0, 1, 2, \ldots;$

d) $\lim\limits_{k \to \infty} E_k = F.$

The properties a), b), c) are obvious and the property d) follows from:

$$\| T_4(s_j)\, x \| = \| T_3^{-1}(t)\, T_3(t)\, T_4(s_j)\, x \| \leqslant \| T_3^{-1}(t) \| \cdot \| T_3(t)\, T_4(s_j)\, x \| \to \infty \qquad (j \to \infty)$$

because of (14).

From d) it follows $mE_k \to mF$, i.e. there is a natural number k_0 such that

$$mE_0 > mF - a > b - 4a \qquad (E_{k_0} = E_0;\ s_{k_0} = s_0) \tag{16}$$

Since

$$\{ E_0 \setminus (s_0 + E_0) \} \subseteq \{ s_0 + ([0, b] \setminus E_0) \}$$

we have

$$m \{ E_0 \setminus (s_0 + E_0) \} \leqslant b - mE_0 < 4a.$$

Further

$$E_0 = \{ E_0 \setminus (s_0 + E_0) \} \cup \{ E_0 \cap (s_0 + E_0) \}$$

and

$$\{ E_0 \setminus (s_0 + E_0) \} \cap \{ E_0 \cap (s_0 + E_0) \} = \emptyset$$

leads to

$$m \{ E_0 \cap (s_0 + E_0) \} > b - 8a > \frac{b}{2}\,. \tag{17}$$

Now (17), (16) and the choice of b imply that the set

$$\{ E_0 \cap (E_0 + s_0) \} \cap \{ E_0 - s_0 \} \tag{18}$$

is not empty.

Hence there is a number t_0 such that t_0, $t_0 + s_0$, $t_0 - s_0$ belong to $E_0 \cap F$. For so chosen number t_0 we have

$$(M^2+1) \| x \| \leqslant \| T_3(t_0)\, T_4(s_0)\, x \| =$$

$$= \| T_1(t_0+s_0)\, T_2(t_0-s_0)\, x \| \leqslant \| T_1(t_0+s_0) \| \cdot \| T_2(t_0-s_0) \| \cdot \| x \| \leqslant M^2 \| x \|$$

which contradicts $x \neq 0$.

Thus $t \to T_4(t)\, x$ is bounded on $[o, a]$. In the same way one proves that this function is bounded on $[-a, o]$. Since a is arbitrary positive number we find that $t \to T_4(t)$ is locally bounded.

From (1) it follows

$$T_2^*(t-s)\, T_1^*(t+s) = T_4^*(s)\, T_3^*(t)$$

from which one can prove that $t \to T_3^*(t)$ is locally bounded. Since $\| T_3^*(t) \| = \| T_3(t) \|$ we find that $t \to T_3(t)$ is locally bounded. From

$$T_1(t) = T_3\left(\frac{t}{2}\right) T_4\left(\frac{t}{2}\right)$$

and

$$T_2(t) = T_3\left(\frac{t}{2}\right) T_4\left(-\frac{t}{2}\right)$$

which one obtains from (1) by replacing t by $\frac{t}{2}$ and s by $\pm\frac{t}{2}$, we conclude that $T_1(t)$ and $T_2(t)$ are also locally bounded functions Q.E.D.

Lemma 2. Under conditions of Theorem 2 the functions $t \to T_i^{-1}(t)$ $(i = 1, 2, 3, 4)$ are locally bounded on R.

Proof. First we prove that $t \to T_3^{-1}(t)$ is bounded on $[o, a]$ for any $a > o$.

We carry the proof in several steps.

I. As in Lemma 1 we take $b > 16a$ and we consider the sets:

$$E_n = \left\{ t \mid t \in [a, b-a];\ \| T_i(t) x \| \geq \frac{1}{n} \| x \| \quad (i = 1, 2, 3, 4) \quad \text{for all } x \in X \right\}^*$$

Now the sets E_n possess the following properties:

a) $E_n \subseteq E_{n+1}$ $\quad (n = 1, 2, \dots)$

b) E_n is measurable and

c) $[a, b-a] = \bigcup_{n=1}^{\infty} E_n$.

There is therefore a set $E_0 = E_{n_0} \subseteq [a, b-a]$ such that

$$m E_0 > b - 3a \text{ and } \| T_i(t) x \| \geq \frac{1}{n_0} \| x \|$$

$$(x \in X,\ t \in E_0,\ i = 1, 2, 3, 4).$$

II. For given t and s with the properties that $t+s$ and $t-s$ belong to E_0 the set

(19) $$\{ T_1(t+s) T_2(t-s) x \mid \| x \| \leq n_0^2 \}$$

contains a sphere $\{ y \mid \| y \| \leq 1, y \in X \}$. Indeed for y, $\| y \| \leq 1$ and for the vector

(20) $$x = T_2^{-1}(t-s) T_1^{-1}(t+s) y$$

holds

$$\| x \| \leq n_0^2,$$

i.e.

$$y = T_1(t+s) T_2(t-s) x$$

belongs to the set (19).

* Notice that $T_i(t)$ is invertible.

III. Now in the same way as in the proof of Lemma 1 one can prove that for any $t \in [0, a]$ there is $s = s(t) \in [a, b-a]$ such that $t+s$ and $t-s$ belong to E_0.
IV. Let $t \in [0, a]$ be given and s be such that $t+s$ and $t-s$ belong to E_0. For y, $\|y\| \leq 1$ by x denote the vector given by (20) and set

$$M = \sup_{0 \leq u \leq a} \| T_4(u) \| .$$

Then

$$\| T_3^{-1}(t) y \| = \| T_4(s) x \| \leq M \| x \| \leq M n_0^2 ,$$

i.e.

$$\| T_3^{-1}(t) \| \leq M n_0^2 .$$

Since M and n_0 do not depend on t we conclude that $T_3^{-1}(t)$ is bounded on the interval $[0, a]$. The rest of Lemma 2 follows in the similar way as in Lemma 1.

Q.E.D.

Lemma 3. Let $t \to T(t)$ be a regular function defined on R with values in the set of all bounded operators on Hilbert space X.

If $t \to T(t)$ is strongly continuous and $t \to T^{-1}(t)$ locally bounded on R, then $t \to T^{-1}(t)$ is also strongly continuous.

Proof. From the assumptions of Lemma 3 it follows

$$\| T^{-1}(t+s) x - T^{-1}(t) x \| = \| T^{-1}(t+s) \{ T(t) - T(t+s) \} T^{-1}(t) x \| \to 0 \quad \text{as} \quad s \to 0 .$$

Q.E.D.

Proof of Theorem 2. Lemma 1 implies that the function $t \to T_4(s) x$ is locally integrable in the Bochner sense for any $x \in X$ ([1], Theorem 3.7.4, p. 80). For $a, b \in R$ set:

$$y_{ab} = \int_a^b T_4(s) x \, ds \tag{21}$$

and denote

$$Y = \{ y_{ab} \mid x \in X ; \ a, b \in R \} .$$

We assert that Y is dense in X. Indeed, suppose that $z \in X$ is orthogonal on Y, i.e.

$$(y_{ab}, z) = \int_a^b (T_4(s) x, z) \, ds = 0$$

for all $x \in X$ and $a, b \in R$. This implies: $(T_4(s) x, z) = 0$ almost everywhere, i.e. for all $s \notin S(x)$, where $S(x) \subseteq R$ is a null-set. Let $e_1, e_2, \ldots$ be a countable and dense set in X and $S = \bigcup_{k=1}^{\infty} S(e_k)$. Since S is null-set we can find $t_0 \notin S$ such that z is orthogonal on $\{ T_4(t_0) x \mid x \in X \}$. This together with the regularity of $T_4(t_0)$ implies $z = 0$, i.e. Y is dense in X.

From this we conclude that for the continuity of T_3 it is sufficient to prove that $T_3(t)$ is continuous for $y \in Y$. For $y_{ab} \in Y$, which is given by (21) we have:

$$\| T_3(t) y_{ab} - T_3(t_0) y_{ab} \| = \left\| \int_a^b [T_3(t) T_4(s) x - T_3(t_0) T_4(s) x] \, ds \right\| =$$

$$= \left\| \int_a^b [T_1(t+s) T_2(t-s) x - T_1(t_0+s) T_2(t_0-s) x] \, ds \right\| =$$

$$= \left\| \int_{a+t}^{b+t} T_1(s) T_2(2t-s) x \, ds - \int_{a+t_0}^{b+t_0} T_1(s) T_2(2t_0-s) x \, ds \right\|$$

$$\leqslant \left\| \int_{a+t}^{a+t_0} T_1(s) T_2(2t-s) x \, ds \right\| + \left\| \int_{b+t_0}^{b+t} T_1(s) T_2(2t-s) x \, ds \right\| +$$

$$+ \left\| \int_{a+t_0}^{b+t_0} T_1(s) [T_2(2t-s) - T_2(2t_0-s)] x \, ds \right\| .$$

Since $T_1(t)$ and $T_2(t)$ are locally bounded, the numbers:

$$\left\| \int_{a+t}^{a+t_0} T_1(s) T_2(2t-s) x \, ds \right\| \quad \text{and} \quad \left\| \int_{b+t_0}^{b+t} T_1(s) T_2(2t-s) x \, ds \right\|$$

tend to zero as $t \to t_0$.

Further we have:

$$\left\| \int_{a+t_0}^{b+t_0} T_1(s) [T_2(2t-s) - T_2(2t_0-s)] x \, ds \right\|$$

$$\leqslant M \int_{a+t_0}^{b+t_0} \| T_2(2t-s) x - T_2(2t_0-s) x \| \, ds,$$

where M is such that $\| T_1(t) \| \leqslant M$ for $t \in [a+t_0, b+t_0]$. On the other hand it is well known that the last integral tends to zero as $t \to t_0$ ([1], Theorem 3.8.3, p. 86). Thus the strong continuity of $T_3(t)$ is proved.

If in (1) we set $u = t+s$ and $v = t-s$ we get

$$T_3\left(\frac{u+v}{2}\right) T_4\left(\frac{u-v}{2}\right) = T_1(u) T_2(v) . \tag{22}$$

The strong continuity of $T_1(t)$ follows from (22) in the same way as the strong continuity of $T_3(t)$. Now, the strong continuity of T_2 and T_4 follows from Lemma 3 and

$$T_2(t) = T_1^{-1}(t) T_3(t)$$

$$T_4(t) = T_3^{-1}(t) T_1(2t).$$

Thus Theorem 2 is proved.

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