PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 4 (18), 1964, p. 157–163.

# ON IDEMPOTENT OPERATORS IN A HILBERT SPACE

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(Presented 17. Avril 1964)

A bounded operator $F$ acting in a Hilbert space $H$ is idempotent if $F^2=F$. Since this equality implies $\|F\|^2 \geqq \|F\|$, we have $\|F\| \geqq 1$ if $F \neq 0$. It is well known that an idempotent $F \neq 0$ is a projection (a self adjoint idempotent) if and only if $\|F\|=1$.

Denote by $H_1=\{x \in H,\ Fx=x\}$ the range and by $H_2=\{x \in H,\ Fx=0\}$ the null space of $F$. The sets $H_1$ and $H_2$ are closed subspaces of $H$. Let $P$ and $Q$ be the projections onto $H_1$ and $H_2$. Since the equality $x=Fx+(I-F)x$, where $I$ is the identity operator, implies that any vector $x \in H$ can be written in the form $x=y+z$, where $y=Fx \in H_1$ and $z=(I-F)x \in H_2$, the Hilbert space $H$ is a direct sum of the subspaces $H_1$ and $H_2$. Therefore, $P \cap Q=0$ and $P \cup Q=I$.

The adjoint operator $F^*$ is also idempotent. Its range is the subspace $H_1^*=H_2^{\perp}$, where $H_2^{\perp}$ denotes the orthogonal complement of $H_2$, and its null space $H_2^*=H_1^{\perp}$. Hence, the corresponding projections are $I-Q$ and $I-P$.

In this paper we shall study in some detail the connections which exist between an idempotent operator $F$, its projections $P$, $Q$, and the products $FF^*$, $F^*F$.

**1.** Let $P$ and $Q$ be two projections. We can ask whether there exists an idempotent operator $F$ having $P$ and $Q$ for its projections. The answer to this question is as follows:

Theorem 1. *$P$ and $Q$ are the projections onto range and null space of a bounded idempotent operator $F$ if and only if $P \cup Q=I$ and $\|PQ\|<1$. In this case we have*

$$F=(I-PQP)^{-1}(P-PQ). \tag{1}$$

*Remark.* The condition $\|PQ\|<1$ implies $\|PQP\|<1$. Consequently, the operator $(I-PQP)^{-1}$ exists and is bounded.

*Proof.* Let $P$ and $Q$ be the projections onto range and null space of a bounded idempotent operator $F$. For any $x \in H$ the vectors $PQx$ and $(I-P)Qx$ are orthogonal. Therefore $\|PQx\|^2+\|(I-P)Qx\|^2=\|Qx\|^2$ and

$$\|PQx\|^2=\|Qx\|^2-\|Qx-PQx\|^2 \leqq \|x\|^2-\|Qx-PQx\|^2. \tag{2}$$

Since $Qx \in H_2$ and $PQx \in H_1$, we have $F(Qx-PQx) = -PQx$. Hence, $\| PQx \| \leq \leq \| F \| \| Qx-PQx \|$. From inequality (2) we now obtain

$$\| PQx \|^2 \leq \| x \|^2 - \frac{\| PQx \|^2}{\| F \|^2}$$

whence

$$\| PQx \|^2 \leq \frac{\| F \|^2 \| x \|^2}{1+\| F \|^2};$$

therefore $\| PQ \| \leq \| F \| / \sqrt{1+\| F \|^2} < 1$. Since we already know that $P \cup Q = I$, the conditions stated in theorem 1 are necessary.

Conversely, let us suppose that $P$ and $Q$ are projections such that $P \cup Q = I$ and $\| PQ \| < 1$. The last condition obviously implies that $P \cap Q = 0$. If an idempotent operator $F$ with the projections $P$ and $Q$ exists, then any vector $x \in H$ can be written in the form $x = y + z$, where $y = Fx \in H_1$ and $z = (I-F)x \in H_2$. Since $Py = y$ and $Qz = z$, we have

$$Px = Py + Pz = y + Pz \tag{3}$$

and

$$Qx = Qy + Qz = Qy + z. \tag{4}$$

From the last equality we obtain $z = Qx - Qy = Qx - QPy$. Substituting this expression for $z$ in the equality (3) we get $Px = y + PQx - PQPy$, i.e.

$$(I-PQP)y = Px - PQx.$$

Now, the inequality $\| PQ \| < 1$ implies the existence of $(I-PQP)^{-1}$, and therefore,

$$y = (I-PQP)^{-1}(Px-PQx), \quad z = Qx - Q(I-PQP)^{-1}(Px-PQx).$$

If we put

$$F = (I-PQP)^{-1}(P-PQ), \tag{5}$$

then $y = Fx$ and $z = Q(I-F)x$. The operator $F$ is idempotent. In fact, since $P$ commutes with $I-PQP$, it commutes also with the inverse $(I-PQP)^{-1}$. Hence $F = P(I-PQP)^{-1}(P-PQ)$ and

$$F^2 = (I-PQP)^{-1}(P-PQ)P(I-PQP)^{-1}(P-PQ) = P(I-PQP)^{-1}(P-PQ) = F.$$

Moreover, it is easily seen that the following relations are valid

$$FP = P, \quad PF = F, \quad FQ = 0. \tag{6}$$

Denote by $H_1'$ and $H_2'$ the range and the null space of $F$. The two first relations (6) imply that $H_1 \subset H_1'$ and $H_1' \subset H_1$. Hence, $H_1' = H_1$. From the third relation we have $H_2 \subset H_2'$. Let now be $x \in H_2'$, thus $Fx = 0$. From (5) one obtains: $Px - PQx = 0$. If we put $y = x - Qx$, we have $Py = 0$ and $Qy = 0$. Since $P \cup Q = I$, it follows that $y = 0$, i.e. $Qx = x$ and $x \in H_2$. Hence, $H_2' = H_2$. The idempotent operator $F$ defined by (5) has $P$ and $Q$ for its projections. Theorem 1 is therefore proved.

**2.** Let us consider the products $FF^* = A$ and $F^*F = B$. The operators $A$ and $B$ are self adjoint and positive. Since $A^2 - A = (FF^* - F)(FF^* - F)^* \geq 0$, the open interval $(0, 1)$ belongs to the resolvent set of $A$. The same of course is true of $B$. If $\| A \| = 1$, then $A^2 - A \leq 0$, therefore $A^2 - A = 0$ and $FF^* = F$, so that $F$ is a projection in this case.

If we take for $F$ the expression (1) we get

$$A=(I-PQP)^{-1}P\,,\quad B=(QP-P)(I-PQP)^{-2}(P-PQ)\,.$$

Let us compute the products $ABA$ and $BAB$:

$$ABA=(FF^*)(F^*F)(FF^*)=FF^*FF^*=A^2.$$

Similarly $BAB=B^2$. These relations are characteristic of self adjoint operators $A$ and $B$ which can be written as the products $FF^*$ and $F^*F$, where $F$ is an idempotent operator. We have namely the following:

Theorem 2. *The equations*

$$(7)\qquad FF^*=A,\quad F^*F=B$$

*can be solved with an idempotent operator $F$ if and only if $A$ and $B$ are self adjoint operators satisfying the relations*

$$(8)\qquad ABA=A^2,\quad BAB=B^2.$$

*$F$ is uniquely determined.*

*Proof.* We already know that the conditions (8) are necessary. Let now $A$ and $B$ be any self adjoint operators satisfying (8) and acting in a Hilbert space $H$. Denote by $H_1^\perp$ the null space of $A$, $H_1^\perp=\{x\in H,\ Ax=0\}$, and by $H_2$ the null space of $B$, $H_2=\{x\in H,\ Bx=0\}$. The set $H_2$ and the orthogonal complement $H_1$ of $H_1^\perp$ are closed subspaces of $H$. Let $P$, $Q$, be the corresponding projections. Then we have

$$(9)\qquad AP=PA=A,\quad BQ=QB=0$$

Since $(ABA)x=A^2x$, i.e. $A(BA-A)x=0$, the vector $(BA-A)x$ belongs to $H_1^\perp$. Hence $P(BA-A)x=0$ or $PBA-A=0$. Taking adjoints we get also $ABP-A=0$. Therefore $A(BPx-x)=0$ for any $x\in H$, consequently $BPx-x\in H_1^\perp$, i.e. $PBPx-$ $-Px=0$. Hence

$$(10)\qquad PBP=P\,.$$

In the same manner we obtain

$$(10^*)\qquad (I-Q)A(I-Q)=I-Q\,.$$

If self adjoint operators $A$ and $B$ satisfy relations (8), then they are $\geqq 0$. In fact, from (8) we deduce

$$A^3=AA^2=A^2BA=ABABA=AB^2A=(AB)(AB)^*\geqq 0.$$

Since $A$ is self adjoint, this inequality implies $A\geqq 0$. Similarly $B\geqq 0$. Now, from (8), (9) and (10) we deduce

$$(11)\qquad PB^2P=PBABP=PBPAPBP=PAP=A.$$

In the same way we get from (10*)

$$(11^*)\qquad (I-Q)A^2(I-Q)=B\,.$$

Since $P$ as a projection is positive, we conclude that $BPB\geqq 0$ and, by computing $(BPB)^2$ we get:

$$(BPB)^2=BPB^2PB=BAB=B^2.$$

Yet, each positive operator has a unique positive square root. Hence, $BPB=B$. Similarly we get $A(I-Q)A=A$.

Now, let us define the operator $F$ by

(12) $$F=PB, \quad \text{thus} \quad F^*=BP$$

Since $F^2=PBPB=PB=F$, the operator $F$ is idempotent. Also $FF^*=PB^2P=A$ and $F^*F=BPB=B$. We have found a solution of equations (7). The conditions (8) are therefore sufficient.

This solution of (7) is unique. For, suppose $F$ and $G$ are idempotent operators such that

$$GG^*=FF^*, \quad G^*G=F^*F,$$

then $G=F$. In fact, since

$$G^*G(I-F)=F^*F(I-F)=0,$$

we have $G(I-F)=0$, thus $GF=G$. Similarly the first equality yields $F^*G^*=F^*$. Taking adjoints we have $GF=F$. Hence $G=F$. This completes the proof of theorem 2.

The previous proof of the existence of $F$ was geometric, but we can also prove this fact algebraically. Let, therefore $\mathfrak{B}$ be a $C^*$-algebra with an identity $I$. *If $A$ and $B$ are self adjoint elements of $\mathfrak{B}$ satisfying relations* (8), *then there exists an idempotent element $F\in\mathfrak{B}$ such that $FF^*=A$ and $F^*F=B$.*

The proof of this statement runs as follows: First, from (8) we obtain the relations

(13) $$AB^n=A^nB, \quad n=1,2,3,\ldots$$

In fact, (11) is an identity for $n=1$. If this equality holds for some $n\geqq 1$, then

$$AB^{n+1}=A^nBB=A^nBAB=A^{n-1}ABAB=A^{n+1}B.$$

Hence, (13) is true also of $n+1$, and therefore, it holds for each $n\geqq 1$.

Next, we have

(14) $$AB^nA=A^{n+1}, \quad BA^nB=B^{n+1}.$$

These equalities follow immediately from (8) and (13).

We already know that equations (8) imply $A\geqq 0$ and $B\geqq 0$. From (14) we get $A(B-I)^2A=A^3-A^2$ and $B(A-I)^2B=B^3-B^2$. Since $A(B-I)^2A\geqq 0$ and $B(A-I)^2B\geqq 0$, it follows $A^3-A^2\geqq 0$ and $B^3-B^2\geqq 0$. These inequalities imply that the open interval (0, 1) lies in the resolvent set of $A$ and of $B$.

If $A=0$, then $B=0$. In this case we have $F=0$. Now, let us suppose $A\neq 0$. It follows from (14) that $\|A\|^{n+1}\leqq\|B\|^n\|A\|^2$, thus $\|A\|^{n-1}\leqq\|B\|^n$. Since this holds for all $n\geqq 1$, we have $\|A\|\leqq\|B\|$. Similarly we get $\|B\|\leqq\|A\|$. Hence $\|A\|=\|B\|\geqq 1$. If we put $\varepsilon=1/\|A\|=1/\|B\|$, then $0<\varepsilon\leqq 1$. Let us now consider the sequences

(15) $$R_n=(I-\varepsilon A)^n, \quad S_n=(I-\varepsilon B)^n.$$

The difference of any two consecutive members is

$$R_n-R_{n+1}=\varepsilon A(I-\varepsilon A)^n, \quad S_n-S_{n+1}=\varepsilon B(I-\varepsilon B)^n.$$

An easy computation shows that the spectrum of the products $\varepsilon A(I-\varepsilon A)^n$ and $\varepsilon B(I-\varepsilon B)^n$ lies in the interval $(0,(1-\varepsilon)^n)$ if $n$ is sufficiently large. It follows that $\|R_{n+1}-R_n\|\leqq(1-\varepsilon)^n$ and $\|S_{n+1}-S_n\|\leqq(1-\varepsilon)^n$. Therefore, the sequences $R_n$ and $S_n$ are convergent. Let $R=\lim R_n$, $S=\lim S_n$ be their limits, then $R, S\in\mathfrak{B}$. Since $R_n^2=R_{2n}$, $S_n^2=S_{2n}$, we have $R^2=R$, $S^2=S$. Thus, $R$ and $S$ are projections.

In the same way we can establish that the sequences $\{AR_n\}$ and $\{BS_n\}$ converge to zero, so that $AR=RA=0$ and $BS=SB=0$. Furthemore, it is not difficult to deduce from (13) and (14) that the relations

$$B(I-R_n)=(I-S_n)A, \quad A(I-S_n)A=A(I-R_n), \quad B(I-R_n)B=B(I-S_n)$$

hold for all $n \geqq 1$. Taking here the limites we find

$$B(I-R)=(I-S)A, \; A(I-S)A=A(I-R)=A, \quad B(I-R)B=B(I-S)=B.$$

Let us define the operator $F \in \mathfrak{B}$ by

(16) $$F=A(I-S)$$

$F$ is idempotent. In fact,

$$F^2=A(I-S)A(I-S)=A(I-S)=F.$$

Since $F^*=(I-S)A$, we have also

$$FF^*=A(I-S)^2A=A, \quad F^*F=(I-S)A^2(I-S)=B(I-R)^2B=B.$$

Therefore, $F$ is a solution of equations (7) and belongs to the algebra $\mathfrak{B}$.

**3.** In the previous section we studied the equations (7) and found the conditions which must be fulfilled by $A$ and $B$ so that (7) is solvable by an idempotent operator $F$. Now, let us consider the first equation only

(17) $$FF^*=A.$$

The question arises whether there is an idempotent operator $F$ satisfying this equation. Obviously, $A$ must be positive and self adjoint. Assume first that $A=\eta P$, where $P$ is a projection and the real number $\eta \geqq 1$. If $\eta=1$, then clearly $F=A=P$, because $FF^*=P=A$. Now let be $\eta>1$, and suppose that a solution $F$ of (17) exists and that $P_1$, $Q$ are its projections. Since

$$FF^*=(I-P_1QP_1)^{-1}P_1 \text{ and } FF^*=A=\eta P,$$

the following equation must hold

(18) $$P_1=\eta P-\eta PP_1QP_1.$$

Multiplication by $P$ on the left yields $PP_1=\eta P-\eta PP_1QP_1=P_1$, hence $P_1=\eta P-\eta P_1QP_1$. If we multiply this by $P_1$, we get $P_1=\eta P_1-\eta P_1QP_1$. It follows $P_1=P$. The equation (18) can therefore be written in the form

(18*) $$PQP=\frac{\eta-1}{\eta}P.$$

Now, let us consider the operator

(19) $$U=\sqrt{\eta-1}\,P-\frac{\eta}{\sqrt{\eta-1}}QP$$

Taking into account (18*) we find $U^*U=P$. Hence, $U$ is a partially isometric operator and, consequently, the product $UU^*$ is also a projection. The subspaces of $H$ corresponding to $P$ and $UU^*$ have the same dimension. Since $PU=0$, the projection $UU^* \leqq I-P$. Hence, denoting by dim $E$ the dimension of the range of a projection $E$, $\dim P \leqq \dim(I-P)$. Since $A=\eta P$ and $\eta>1$, the range of the projection $P$ is the subspace where $A>1$.

This partial result can be generalized as follows:

Theorem 3. *Let $A$ be a bounded self adjoint operator such that $A^3 - A^2 \geqq 0$ and let $E_\lambda$ be its spectral resolution of the identity, then the equation* (17) *is solvable by an idempotent operator $F$ if and only if* $\dim (I - E_1) \leqq \dim E_0$.

*Proof.* a) First, let us suppose that there exits an idempotent operator $F$ satisfying (17) and let $F = (I - PQP)^{-1}(P - PQ)$, where $P$ and $Q$ are the projections of $F$. Then $A = (I - PQP)^{-1} P$. We already know that $A \geqq 0$ and $A^2 - A \geqq 0$, thus $A^3 - A^2 \geqq 0$.

Let us now consider the operator $T = QP$ and let

$$T = V | T |, \quad T^* = | T | V^*$$

be its canonical decomposition, where $V$ is a partially isometric operator and $|T| = (T^*T)^{\frac{1}{2}} = (PQP)^{\frac{1}{2}}$. If we denote by $N$ the null space of $T$ and by $N^*$ the null space of $T^*$, thus $N = \{x \in H, \ Tx = 0\}$ and $N^* = \{x \in H, \ T^*x = 0\}$, then the base space of $V$ is the subspace $H \ominus N$ and the range of $V$ is $H \ominus N^*$. The projections onto $N$ and $N^*$ are $(I - P) + P \cap (I - Q)$ and $(I - Q) + Q \cap (I - P)$, therefore

$$E = V^*V = P - P \cap (I - Q) \leqq P, \quad E' = VV^* = Q - Q \cap (I - P) \leqq Q.$$

We have also $E | T | = V^*V | T | = | T |$ and $P | T | = P (PQP)^{\frac{1}{2}} = (PQP)^{\frac{1}{2}} = |T|$. Let us consider the product $PV$. First, we have $PV | T | = PT = PQP = | T |^2$. Next,

$$(PV - | T |)(V^*P - | T |) = P(VV^*)P - PV | T | - | T | V^*P + | T |^2 = P(VV^*)P - | T |^2$$

Since the inequality $VV^* \leqq Q$ implies $P(VV^*)P \leqq PQP = | T |^2$, we have

$$(PV - | T |)(PV - | T |)^* \leqq 0.$$

It follows that $PV = | T |$.

Let us define $U$ by

$$U = (| T | - V)(I - | T |^2)^{-\frac{1}{2}}$$

$U$ is a bounded operator and such that $PU = 0$. Since $| T | V = | T | PV = | T |^2$ we get

$$(| T | - V)^* (| T | - V) = E - | T |^2 = E(I - | T |^2)$$

and therefore $U^*U = E$. Hence, $U$ is a partially isometric opeator, consequently, the product $UU^*$ is also a projection. Because $PU = 0$, it is $UU^* \leqq I - P$. The subspaces of $H$ corresponding to $E$ and $UU^*$ are isomorphic, have thus the same dimension.

Let $E_\lambda$ be the spectral function of $A$. We know that $A \geqq 0$ and that the open interval $(0, 1)$ belongs to the resolvent set of $A$; hence $E_\lambda = 0$ if $\lambda = 0$ and $E_0 = E_{1-0}$. It follows that $E_1 = E_0 + (E_1 - E_{1-0})$ is the projection onto the null space of $A^2 - A$. On the other hand, the equality

$$A^2 - A = (I - PQP)^{-2}(PQP) = (I - | T |^2)^{-2} | T |^2$$

implies that the operators $A^2 - A$ and $| T |$ have the same null space. Since the null space $N$ of $T$ coincides with that of $| T |$, we conclude that $E_1$ is the projection onto $N$. This implies $E=I-E_1$. Now, it follows from $E_0=I-P$, $U^*U=E$ and $UU^*\leqq I-P$, that $\dim(I-E_1)=\dim E=\dim UU^*\leqq\dim E_0$. The conditions of theorem 3 are thus necessary.

b) Conversely, let us suppose that a self adjoint operator $A$ satisfies these conditions. First, we infer from $A^3-A^2\geqq 0$ that $E_\lambda=0$ for $\lambda<0$ and $E_0=E_{1-0}$, where $E_\lambda$ is the spectral function of $A$. If we put $P=I-E_0$, we have $PA=AP=A$ and from $A-P=\int_{1-0}^{\infty}(\lambda-1)\,dE_\lambda$ we conclude that $A-P\geqq 0$. Thus $(A-P)^{\frac{1}{2}}$ exists. Moreover, because $\dim(I-E_1)\leqq\dim E_0$, there is a partially isometric operator $U$ such that $U^*U=I-E_1$ and $UU^*\leqq I-P=E_0$. Hence $PU=0$. Let $\varphi(\lambda)$ be any function of the real variable $\lambda$, measurable for $\lambda\geqq 0$ and such that $|\varphi(\lambda)|=1$. Define the operator $F$ as follows

$$F=P-\varphi(A)(A-P)^{\frac{1}{2}}U^*.$$

$\varphi(A)$ is a unitary operator which commutes with $P$. Since $PU=U^*P=0$ and $(A-P)^{\frac{1}{2}}=P(A-P)^{\frac{1}{2}}$, we have $F^2=F$, so that $F$ is idempotent. On the other hand,

$$F^*=P-U(A-P)^{\frac{1}{2}}\overline{\varphi}(A)$$

An easy computation gives $FF^*=A$. Thus, we have found an idempotent operator $F$ satisfying (17). The conditions of theorem 3 are therefore also sufficient.