PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 4 (18), 1964, p. 107—108.

REMARK ON RECENT TWO RESULTS OF DILWORTH AND GLEASON

Djuro Kurepa

(Presented January 24, 1964.)

In a recent paper, Gleason-Dilworth [1], one reads following theorems.

1. Theorem A. *If O is any (partially or totally) ordered set and IO is the set of all order ideals of O, then $(IO, \subset)$ is not order isomorphic to any subset of $(O, <)$.*

2. Theorem B. *Let $(O, <)$ be a (partially or totally) ordered set; if φ is any one-to-one map of $(IO, \subset)$ into $(O, <)$ then neither φ nor φ^{-1} is order preserving.*

3. Now, the theorem A and the part $B(\varphi)$ of theorem B concerning φ *are corollaries of the theorem* 1 *in my paper* [2] *stating the following.*

3.1. Theorem. *There is no strictly increasing mapping of $(wO, \dashv)$ into $(O, <)$; where one has the following definition.*

3.2. Definition. For any ordered set $(O, <)$ let wO denote the set of all the well ordered subsets of $(O, <)$, the empty set included and the

3.3. relation $A \dashv B$ means that A is a *proper initial section of* B (i.e. A is a proper ideal of B); we write $=\!\!|$ for $\dashv$ or $=$.

The forgoing theorem 3.1 implies the following two corollaries.

3.4. Corollary. *There is no strictly increasing mapping of*

$$(I_w O, \subset) \text{ into } (O, <) \text{where}$$

3.5. $$I_w O = \{X_I;\ X \in wO\} \text{ and}$$

3.6. $$X_I = \bigcup_{x \in X} O(\cdot, x],\ \ O(\cdot, x] = \{y;\ y \in O, y \leqslant x\}.$$

And the corollary 3.4. implies even a stronger result then the Theorem A obtained from A by *considering instead of the set IO of all ideals only the part $I_I O$ of all ideals each of which is cofinal to a chain of $(O, <)$.*

The corollary 3.4 implies also the stronger result then $B(\varphi)$ obtained by substitution $IO \rightarrow I_w O$:

3.7. Theorem B. *If φ[1] is any mapping of $(I_w O, \subset)$ into $(O, <)$, then φ is not strictly increasing.*

[1] φ need not be supposed to be one-to-one.

3.8. The corollary 3.4 follows from the theorem 3.1 by observing that any strictly increasing mapping f from $(I_w O, \subset)$ into $(O, <)$ would yield the following strictly increasing mapping g from $(wO, \dashv)$ into $(O, <)$:

For any $X \in wO$ let $gX = fX_I$, in particular $g\emptyset = f\emptyset$.

As a matter of fact the mapping

$$(1) \qquad X \to X_I, \quad (X \in (wO, \dashv))$$

is strictly increasing.

If then there were a strictly increasing mapping

$$(2) \qquad (I_w(O, <), \subset) \ni y \to fy \in (O, <),$$

then the composition of the strictly increasing mappings (1), (2) would yield a strictly increasing mapping on $(w(O, <), \subset)$ into $(O, <)$, contradicting the statement 3.1.

4. *If in the wording of the theorem B one replaces the set $(IO, \subset)$ by the set $(I_w O, \subset)$, one might have a false result.*

Such a situation occurs e.g. if $(O, <)$ is any infinite antichain $(A, <)$; namely, if $a \in A$, then there is a one-to-one mapping h on A onto $A \setminus \{a\}$; if then we put $\varphi\emptyset = a$ and $\varphi\{x\} = hx$ for every $x \in A$, the mapping φ is one-to-one and the mapping φ^{-1} is order preserving because every non empty chain in $(A, <)$ has just one point.

Such a situation does [does not] occur every time when $I_w(O, <)$ has the same cardinality as an antichain [chain] A [resp. L] of $(O, <)$: any one-to-one mapping of I_w into A is such that φ^{-1} is [is not] increasing. The text corresponding to [ ] is a direct consequence of the theorem 3.1.

BIBLIOGRAPHY

1. Gleason A. M — Dilworth R. P., *A generalized Cantor theorem*, Proc. Amer. Math. Soc. 13 (1962), 704—705.

2. Kurepa Đ., *Ensembles ordonnés et leurs sous-ensembles bien ordonnés*, C. r. Acad. Sci. Paris 242 (1956), 2202—2203.