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SOME REFLEXIONS ON SETS AND NON-SETS

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1. Membership relation. 1.0. The relation $\in$ meaning *"to be an element of"* was introduced as late as at the end of the 19[th] century (G. Peano).

1.1. The relation $\in$ is not characteristic for sets.

1.1.1. One could call granular structure G any thing consisting of members i.e. satisfying the identity

$$G=\{x;\ x\in G\}.$$

The solution x of $x\in G$ may be of a very various character and complexity.

1.1.2. There are sets S such that $x\in S$ has no solution x; such a set is the empty set $\emptyset$; one convenes that $\emptyset$ is unique; but one might consider a theory of sets in which there are many void sets.

1.1.3. There are non-sets X consisting of all the x satisfying $x\in X$; such a thing are e.g.: the class O of all ordinal numbers, the class KO of all cardinal-ordinal numbers i.e. of all ordinals $<\omega$ and of all ordinal numbers of the form ω_α $(\alpha\in O)$, the class K of all the cardinal numbers, the class of all sets, the hypertree $(P, \dashv)$ consisting of all the sequences

$$s\cdot\cdot\cdot s_0,\ s_1,\ \ldots,\ s_{\alpha'},\ \ldots\quad (\alpha'<\alpha,\ \alpha\in O)$$

such that $s_{\alpha,}$ is an ordinal number satisfying

$$s_{\alpha'}<\omega_{[\alpha']}\,;\ \omega_{[\alpha']}$$

denotes the first ordinal number such that the cardinal numbers which are $<k\,\omega_{(\alpha')}$ form a well ordered set of type α'; for sequences s, t one denotes $s=\!\!|\,t$, provided s be an initial section of t; if $s=\!\!|\,t$ and $s\neq t$ one writes $a\dashv b$. The *hypertree* $(P, \dashv)$ is connected to permutations of sets and of numbers.

1.2. For sets the binary ε-relation is antireflexive, antisymmetric and antitransitive. In this sense the relation ε is complete negation of equivalence relations (which are: reflexive, symmetric and transitive). Consequently, the theory of sets being based on the theory of ε-relation, is in a particular connexion to the theory of equivalence relations in the frame of the theory of sets itself.

1.3. The property of being a member (element), or class is of a relative character.

1.4. It is interesting that for any thing b — no matter whether b is a set or a non set — one might let correspond the set $\{b\}$ consisting of b as the unique member (cf. § 7.3).

1.5. Repetition sets or spectra.

1.5.1. For any set S and any member s of S one does not allow the relation $s \in S \setminus \{s\}$. Therefore e.g. $\{2, 3, 3, 3, 4, 4\} \setminus \{3\} = \{2, 4\}$.

1.5.2. On the other hand, there are structures in which it is relevant whether a member occurs once, twice or several times. E.g. for an algebraic polynomial $a(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, a_n \neq 0$, one considers not only the set σ_a of zeros of a but also the unordered sequence

$$Sa = a_{(1)}, a_{(2)}, \ldots, a_{(n)}$$

of all the zeros of $a(x)$, each with the corresponding frequency and such that

$$a(x) = (x - a_{(1)})(x - a_{(2)}) \cdots (x - a_{(n)}) a_n,$$

where n is the *degree* of $a(x)$.

1.5.3. The whole theory of "*sets with repetition*" or of *R-sets* could be built; where *R-set or spectrum* is any ordered pair (S, f) of a set S and a mapping $f: S \to K$ such that for every $x \in S$ the symbol fx denotes a cardinal number > 0 indicating the frequency of x in S. In such *R-sets* the relation $x \in S \setminus \{x\}$ is well allowed.

2. On the *i*-operators. 2.1. For every object b let $\{b\}$ or ib denote the set consisting of b as its unique term (cf. 1.4). 2.2. The opposite operator: the *anti i-operator or $-i$-operator*, associates to every set S all the members of S; thus $-i\{b\} = b$, $-i\{1, 2\} = 1, 2$ etc. The anti $-i-$ operator is a multi-valued function defined on every system of sets, granular structures etc.

2.2. The *i-operator by iteration yields the i^2-operator:*

$$i^2 = i\,i \quad \text{i.e.} \quad i^2 b = i\,ib.$$

For any object b one could set

$$i^0 b = b,\ i^1 b = ib = \{b\},\ i^{(\alpha+1)^*} b = i(i^{\alpha^*} b),\ i^{\lambda^*} b = \cdots ii \cdots ii\,b$$

for every limit ordinal λ.

2.2.1. Example. We have the entities:

$$1,\ i1 = \{1\},\ i^2 i = \{\{1\}\}, \ldots, i^{\omega^*} 1 = \cdots \{\{\{1\}\}\} \cdots$$

The first term 1 is not a set; the last one neither. While 1 has no "elements" and no structure — 1 is an atom — the entity $c = i^{\omega^*} 1$ has a structure; in particular c is a kind of infinitely complex unity.

2.3. Obviously, the objects b; $\{b\} = ib$, $i^2 b = \{\{b\}\}$ are mutually related. We shall say that b is in ε^2-relation to $i^2 b$ and write $b \in^2 i^2 b$. More generally, we shall write $b \in^2 S$, provided $ib \in S$; and for any ordinal number α we shall write $b \in^{(\alpha)} S$, provided $i^{(\alpha-1)^*} b \in S$ or $i^{\alpha^*} \in S$, according as α is of the first or of the second kind. We get in this way the ε-relations:

$$\varepsilon^0 \text{ meaning } =,\ \varepsilon^1 = \varepsilon,\ \varepsilon^2, \ldots, \varepsilon^\alpha, \ldots$$

for every ordinal number α.

2.4. Sub-element relation. The "logical sum" of these relations might be called the sub-element relation and denoted by E; xEy is read: *x is a sub-element of y*; in particular case that E means $\in$, x is an element of y. It is to be observed that a set S can contain a thing x as its element or subelement of various degrees, as it is shown by considering the set

$$\{1, i\,1, i^2\,1, i^3\,1, \ldots\}$$

2.5. *For sets S the objects*

$$i^0 S = S,\ iS,\ i^2 S,\ \ldots,\ i^{\alpha^*} S,\ \ldots \qquad \text{(for any ordinal } \alpha)$$

are pairwise different. For non-sets S, it is conceivable that the foregoing objects are not all pairwise different.

2.6. It is interesting to observe that there are sets S such that if $x \in^2 S$ then $x \in S$; such are the sets

$$S = \{\emptyset, i\,\emptyset, i^2\,\emptyset\},$$

$$\{\emptyset, i\,\emptyset, i^2\,\emptyset, \ldots\},$$

where $\emptyset$ is the empty set.

2.7. For every object b we have the hypersequence

$$i^0 b = b,\ i^1 b,\ \ldots\ i^{\alpha^*} b,\ \ldots$$

of sets $i^{\alpha^*} b$ for $\alpha^- < \alpha$ and of non sets $i^{\alpha^*} b$ for $\alpha^- = \alpha$; as to $i^0 b = b$, b might be a set as well as a non set.

3. Granular structures and non granular structures.

3.1. In every granular structure G the elements of G are differentiated; there are granular structures which are nonsets. Such structure is every class which is "too extensive" to be a set, e.g. the class of all ordinal numbers or the class of all sets.

3.2. A new kind of non-set structure is obtained by considering *things with non differentiated elements*, the "elements" having no individuality (in atomic physics, in biology and in the theory of big molecules one is dealing with such non-differentiated non granular structures).

3.3. Another kind of non-set structures is obtained, when the "evolution" of S is too much put forward in such a way that the constituents of S need not be elements of S. Such one is the structure $i^{\omega^*}\,1 = \cdots\{\{\{1\}\}\}\cdots$: too many shells are present and we are not able to reach from outside any constituent. In this example there is a unique constituent; it is ready to form more complicated structures with many constituents, tied and quite nonseparable mutually.

3.4. The *notion of structure* — granular or non granular — *is very general and multivalent.* The study of various structures is the very object of many human activitities. Every science is a structure. Every machine is a structure. Language is a structure. Mental structures are of vital importance; mathematical structures are reflecting some special observed structures. The classification of various structures, the interconnections between them are very important topics. It is very important to examine the transitions and mutations of a structure moving from a domain in another domain. As example let us mention the following structures: relation, group, system, family, operation etc. which generated in biology but are transplanted in other fields, particularly into mathematics.

4. *Vacuous or void set. All-sets.*

4.1. We assume the existence of a set without elements or proper parts and being part of every set; it is called the vacuous or empty set and denoted by $\bigcirc$ or v or $\wedge$ or $\emptyset$. Consequently, v is a set but the relation $x \in v$ does not hold. We assume that v is unique. The relation $v \subseteq S$ for every S is really a *definition* of v.

The consequence of the convention $v \subseteq S$ is $v \in PS$ for every set S (as usually, PS denotes the set of all the parts of S).

The set $\{v\}$ is not empty: $\{v\}$ consists of v as its single element.

The philosophical aspect of the distinction of v from $\{v\}$ is evident. The mathematical implications of this distinction are very far-reaching. Is it really non-contradictory to form $\{v\}$ and to distinguish $\{v\}$ and v?

v consists of nothing on the one hand, and on the other hand *v is a part of every set S* and even an *effective element* of every *P-set* PS. Hence, S being any set the vacuous set v is an element and a part of PS i. e. $v \in PS \cap PPS$.

The void set is connected to a number — with 0[1].

The notion of void set is a useful convention and presents a magistral dialectical unification of two different items: void and non void. It is to be observed that void sets were introduced as late as the beginning of this century. There is a unique void set, although by provenience one could classify the void sets in very various ways. The properties, conventions, terminology concerning vacuous set might be very various and in mutual contradiction. For instance, the set (space) v is considered as dense, non dense, nowhere dense, finite, etc.

The considerations about the number 0 form a chapter of the theory of void sets; dynamic theory of 0 is the very basis of infinitesimal processes.

4.2. The logical counterpart of void set — the "all-set" is not conceivable as a *set* because such an idea would lead to contradictions. The non-vacuous sets are either finite or transfinite. The theory of transfinite sets is of great philosophical importance, and closely tied to logical quantors and hierarchy types. The theory of void set(s) on the one hand and that one of transfinite sets on the other hand are two aspects of human mathematical and philosophical activity.

4.3. *The void set and atoms.* The void set v is to be distinguished from general "*atoms*". Atoms have no elements; e.g. such are the points in the sense of Euclid. Various points are elements of sets. Since the points are distinguishable, we are able to adjoin every individual point p to every set S — the result is again a set, the set $\{p\} \cup S$; in general, this set differs from S; what is to be compared with the identity $S \cup v = S$ for every set S.

5. On the operator $x \cup \{x\} = ux$ for every thing x.

5.1. Definition: *ub is obtained by adjoining to b the thing b as an element* i.e. ub consists of the elements of b and of b as a member:

(1) $$x \in ub \Leftrightarrow x = b \vee x \in b.$$

5.2. Value of ub for any set b. If b is a set, then ub is a set containing as well b as an element as well as all the elements of b; since $b \notin b$ (b being

[1] The role of the number 0 is tremendous. How the role of 0 might be of a relative character, let us remember that 0 assumes the role of neutral element in a group.

If we are dealing with single-valued fonctions f on a set S, we could realize f as changing every x of S into fx; by idealization, we consider the identity *mapping* too as a "*changing*", although there is no changing at all. Similarly, the *resting* is called a *moving* with the speed 0.

a set), we see that both $ub \supset b$ and $ub \ni b$ and more precisely $ub \setminus \{b\} \ni b$. In particular, $ub \neq b$ and moreover, $b \neq ub \neq \{b\}$ for every non void set b. One has $uv = v \cup \{v\} = \{v\}$.

5.3. Ub for any non set *b*. Let us consider the case when b. is not a set.

5.3.1. Case: *b has no element(s)*: the relation $x \in b$ is not possible. In this case $ub = \{b\}$. In fact, since the relation $x \in b$ has no solution x, then $x \in ub \Rightarrow x = b$ and consequently $ub = \{b\}$.

5.3.2. Case: *b is a non-set containing at least one element:* $x \in b$ is possible for some x (this case occurs e.g. if b contains very many elements—*b is a class, a superset*). In this case again, ub contains as elements all the elements of b as well as b itself. Only, in this case the relation $b \in b$ is not excluded. If $b \in b$, then $ub = b$ and $b \in ub$. If $b \notin b$, then $ub \ni b$ and $ub \setminus b \ni b$; consequently, $ub \neq b$, $\{b\}$. In particular, for every non void set b we have $b \notin b$ and therefore $ub \neq b$, $\{b\}$, as stated already in § 5.2.

5.4. Theorem. *The system*

$$ub = \{b\}, \; b \text{ non} \in b \tag{1}$$

is characteristic for atoms or points i.e. for things containing no element; in particular, for the vacuous set $\emptyset$ *one has* $u\emptyset = \{\emptyset\}$ (the vacuous set is considered also as an atom). In other words, if (1) holds, then b is an atom; and conversely, if b is an atom then (1) holds.

First, if b is an atom, then (1) holds, as was shown in 5.3.1. Let us prove the converse: (1) implies that b is an atom. In virtue of 5.2. b is not a non vacuous set; consequently, b is either vacuous set $\emptyset$ or b is a non-set; in the last case, there is no x satisfying $x \in b$ i.e. b is an atom. Suppose on the contrary that the relation $x \in b$ be possible. Since b non $\in b$, then b and x would be two different elements of ub, in contradiction to the hypothesis (1) stating that ub is a single-pointset $\{b\}$.

The theorem 5.4. may be expressed in the following form.

5.5. Theorem. *The relation $x \in b$ holds for at least one x if and only if $ub \neq \{b\}$ or $b \in b$.*

Let us prove this theorem directly.

1. First, if $x \in b$ is possible, b is either a non empty set or a superset; if b is a non vacuous set, then $b \neq ub \neq \{b\}$; if b is a superset, then ub also is a superset and might not be equal to the set $\{b\}$ consisting of the single member b. Consequently, $x \in b \Rightarrow ub \neq \{b\}$.

2. Conversely, let us prove that $ub \neq \{b\} \Rightarrow x \in b$ for some x. We have to distinguish two cases.

The implication being obvious for the case $b \in b$, let us suppose that b non $\in b$.

First case: *b is a set;* since $u\emptyset = \{\emptyset\}$, one has necessarily $b \neq \emptyset$ and hence $x \in b$ for some x.

Second case: *b is a non set;* b is not an atom because every atom satisfies $ub = \{b\}$. Consequently, b is a superset and consequently, *one ha* $x \in b$ for some x.

5.6. Theorem. If

(1) $b \cup \{b\} = b$, then $b \in b$; and conversely,

(2) $b \in b$ implies (1); *consequently* (3) $b \cup \{b\} = b \Leftrightarrow b \in b$.

First, by definition $b \in ub$ i.e. $b \in b \cup \{b\}$ and hence by (1) one has $b \in b$. This means that (2) $\Leftrightarrow$ (1).

Secondly, if $b \in b$, then $\{b\} \subseteq b$ and $b \cup \{b\} = b$; in other words (2) $\Rightarrow$ (1).

6. Operator u_α for any ordinal α.

6.1. Definition. Let α be any ordinal $\neq 0$; for every thing b we set $u_1 b = i^0 b = b$

$$u_2 b = b \cup \{b\} = i^0 b \cup i^1 b$$

$$u_3 b = b \cup \{b\} \cup \{\{b\}\} = i^0 b \cup i^1 b \cup i^2 b$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$u_\alpha b = \bigcup i^{\alpha'} b, \ (\alpha' < \alpha) \text{ for every ordinal } \alpha > 0.$$

One could say that

$$u_\alpha = \bigcup_{\alpha'} i^{\alpha'} \qquad (\alpha' < \alpha).$$

In the foregoing sections we considered the operator u_2; the index α in u_α indicates the order type of summands in the definition of u_α.

6.2. Of course, it is a particular problem to study the foregoing functions u_α as well as other ones connected to $i^{\alpha *}$-operators.

6.2.1. It should be particularly interesting to study the function $i^0 \cup i^2$ i.e. to form $b \cup \{\{b\}\}$ for any object b.

6.3. Theorem. *$b \in b \Rightarrow u_\alpha b = b$ as well as $i^\alpha b \in b$ for any positive ordinal $\alpha < \omega$.*

As a matter of fact, we have the following chain of implications:

$$b \in b \Rightarrow \{b\} \subseteq b \Rightarrow ib \subseteq i^0 b \Rightarrow i^2 b \subseteq ib \subseteq i^0 b \Rightarrow ib \in b;$$

from here, by iteration one gets $i^2 b \in b$, $i^3 b \in b$, ...

Therefore we have the following

$$u_2 b = u_0 b \cup u_1 b = b \cup \{b\} = b$$

$$u_3 b = b \cup \{b\} \cup \{\{b\}\} = b$$

$$\cdots\cdots\cdots\cdots$$

$$u_n b = b \text{ and}$$

$$u_\omega b = b \cup ib \cup i^2 b \cup \cdots = b.$$

7. The foregoing considerations show a particular and very important role of the relation

(1) $$b \in b$$

and of the mapping

(2) $$b \to \{b\}.$$

The simplest standpoint is the following one:

7.1. *No set satisfies $b \in b$;*

7.2. *No non-set satisfies $b \to \{b\}$.*

7.3. It is a special task to consider *axiomatically* also such theories of sets in which one of the propositions 7.1, 7.2 or both are not accepted.

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