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ON A CLASS OF FUNCTIONAL EQUATIONS

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The functional equations to be considered are certain generalizations of the functional equation

$$f(x_1+x_2, x_3)-f(x_1, x_2+x_3)+f(x_1, x_2)-f(x_2, x_3)=0 \tag{1}$$

to the case where all the figuring functions are different:

$$F(x_1+x_2, x_3)+G(x_1, x_2+x_3)+H(x_1, x_2)+K(x_2, x_3)=0. \tag{2}$$

D. S. Mitrinović and D. Ž. Đoković [1, 2] have considered the special cases where $H(x, y)$ resp. $-K(x, y)$ resp. $-G(x, y)$ is equal to $F(x, y)$ or $F(y, x)$.

In the present paper we solve the functional equation (2) without any restriction on the functions F, G, H, K. We prove the following

T h e o r e m. Let us consider the functional equation (2) where the variables x_i are elements of an abelian group A and the values of functions F, G, H, K are elements of an other abelian group A'. Then the most general form of the solutions of this equation (2) is

$$\begin{aligned} F(x, y)&=f(x, y)+p(x)+\varphi(y),\\ G(x, y)&=-f(x, y)+\psi(x)+q(y),\\ H(x, y)&=f(x, y)-p(x+y)-\psi(x)+r(y),\\ K(x, y)&=-f(x, y)-q(x+y)-r(x)-\varphi(y), \end{aligned} \tag{3}$$

where $f(x, y)$ is an arbitrary solution of (1) and p, q, r, φ, ψ are arbitrary functions. A function f in the representation (3) can be expressed by F in the following way:

$$f(x, y)=F(x, y)-F(x, 0)-F(0, y)+F(0, 0) \tag{4}$$

and p, q, r, φ, ψ can be given by

$$\begin{aligned} &p(x)=F(x, 0), \qquad \varphi(y)=F(0, y)-F(0, 0),\\ &q(y)=G(0, y), \qquad \psi(x)=G(x, 0)-G(0, 0),\\ &r(x)=F(x, 0)+H(0, x)=-G(0, x)-K(x, 0). \end{aligned} \tag{5}$$

It can be shown by a simple calculation that (3) satisfies (2) if (1) is fulfilled. Thus in order to prove the theorem it is enough to show that any solution of (2) can be written in the form (3) where also (1) holds.

First we express H, K by means of F, G and by certain functions of a single variable. Let us substitute $x_1=0$, resp. $x_3=0$ in (2) then we get

$$\begin{aligned} K(x_2, x_3) &= -F(x_2, x_3) - G(0, x_2+x_3) - H(0, x_2), \\ H(x_1, x_2) &= -G(x_1, x_2) - F(x_1+x_2, 0) - K(x_2, 0). \end{aligned} \tag{6}$$

By this we obtain from (2):

$$F(x_1+x_2, x_3) - F(x_2, x_3) - G(0, x_2+x_3) - H(0, x_2) \\ + G(x_1, x_2+x_3) - G(x_1, x_2) - F(x_1+x_2, 0) - K(x_2, 0) = 0,$$

i. e.

$$\begin{aligned} & F(x_1+x_2, x_3) - F(x_1+x_2, 0) - F(x_2, x_3) + F(x_2, 0) \\ & + G(x_1, x_2+x_3) - G(0, x_2+x_3) - G(x_1, x_2) + G(0, x_2) \\ & - F(x_2, 0) - G(0, x_2) - H(0, x_2) - K(x_2, 0) = 0. \end{aligned} \tag{7}$$

But (2) with $x_1=x_3=0$ gives that

$$F(x_2, 0) + G(0, x_2) + H(0, x_2) + K(x_2, 0) = 0,$$

hence by denoting

$$\Phi(x, y) \overset{\text{def}}{=} F(x, y) - F(x, 0), \quad \Psi(x, y) \overset{\text{def}}{=} G(x, y) - G(0, y) \tag{8}$$

we reduce (7) and thus (2) to a simpler equation:

$$\Phi(x_1+x_2, x_3) - \Phi(x_2, x_3) + \Psi(x_1, x_2+x_3) - \Psi(x_1, x_2) = 0. \tag{9}$$

Now we express Ψ by Φ by putting $x_2=0$ into (9):

$$\Psi(x_1, x_3) = -\Phi(x_1, x_3) + \Phi(0, x_3) + \Psi(x_1, 0). \tag{10}$$

So (9) gives

$$\Phi(x_1+x_2, x_3) - \Phi(x_2, x_3) - \Phi(x_1, x_2+x_3) + \Phi(0, x_2+x_3) \\ + \Psi(x_1, 0) + \Phi(x_1, x_2) - \Phi(0, x_2) - \Psi(x_1, 0) = 0,$$

i. e.

$$\Phi(x_1+x_2, x_3) - \Phi(0, x_3) - [\Phi(x_1, x_2+x_3) - \Phi(0, x_2+x_3)] \\ + \Phi(x_1, x_2) - \Phi(0, x_2) - [\Phi(x_2, x_3) - \Phi(0, x_3)] = 0$$

showing that

$$f(x, y) \overset{\text{def}}{=} \Phi(x, y) - \Phi(0, y) \tag{11}$$

satisfies (1). Taking (11), (8_1), (8_2), (10), (6) into account, F, G, H, K can be expressed by f:

$$\begin{aligned} F(x, y) &= \Phi(x, y) + F(x, 0) = f(x, y) + \Phi(0, y) + F(x, 0), \\ G(x, y) &= \Psi(x, y) + G(0, y) = -\Phi(x, y) + \Phi(0, y) + \Psi(x, 0) + G(0, y) \\ &= -f(x, y) + \Psi(x, 0) + G(0, y), \\ H(x, y) &= f(x, y) - \Psi(x, 0) - G(0, y) - F(x+y, 0) - K(y, 0), \\ K(x, y) &= -f(x, y) - F(x, 0) - \Phi(0, y) - G(0, x+y) - H(0, x). \end{aligned}$$

With the notations

$$
\begin{aligned}
&p(x)=F(x,0), \quad \varphi(y)=\Phi(0,y),\\
&\psi(x)=\Psi(x,0), \quad q(y)=G(0,y),\\
&-r(x)=G(0,x)+K(x,0)=-F(x,0)-H(0,x)
\end{aligned}
\tag{12}
$$

we have (3).

Both the definitions of $r(x)$ under (12) are compatible each with another; this is clear if we put $x_2=x,\ x_1=x_3=0$ in (2).

Finally, the formulae (4)—(5) can be verified if we take (11), (8) resp. (12), (8) into account.

This completes the proof of the Theorem.

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