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A REMARK ON THE PAPER OF I. RAITCHINOV „SUR UN THÉORÈME DE PÓLYA"

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In the first part of this paper I shall answer the question proposed by I. Raitchinov [1]. In the second part I give the simplified proof of Raitchinov's theorem.

1. I. Raitchinov [1] has considered the set M of all linear operators L whose domain is the set S of all polynomials in z with complex coefficients, $L(f)\in S$ for $f\in S$, and which satisfy the additional condition

$$L[f(z)]=g(z) \quad \Rightarrow \quad L[f(z+a)]=g(z+a). \tag{1}$$

In his paper he proposed the following question: Is the set M identical with the set E of all operators of the form

$$L(f)=\sum_{k=0}^{\infty} a_k f^{(k)}(z) \qquad (f\in S)? \tag{2}$$

It is obvious that $E\subset M$. We shall prove that $E\supset M$, so $E=M$.

Let n and m denote the degrees of f and g, respectively. Starting from (1), we find

$$L\left[\sum_{\nu=0}^{n}\frac{a^\nu}{\nu!}f^{(\nu)}(z)\right]=\sum_{\nu=0}^{m}\frac{a^\nu}{\nu!}g^{(\nu)}(z),$$

$$\sum_{\nu=0}^{n}\frac{a^\nu}{\nu!}L[f^{(\nu)}(z)]=\sum_{\nu=0}^{m}\frac{a^\nu}{\nu!}g^{(\nu)}(z).$$

Hence, $n\geqslant m$ and $L[f^{(\nu)}(z)]=g^{(\nu)}(z)$, from which it follows

$$L\left[\int_0^z f(z)\,dz\right]=\int^z L[f(z)]\;dz+\text{const.} \tag{3}$$

Using (3) we find

$$L[1]=a_0,$$

$$L[z]=a_0 z+a_1,$$

$$L\left[\frac{z^2}{2!}\right]=\frac{a_0}{2!} z^2+a_1 z+a_2,$$

$$\vdots$$

$$L\left[\frac{z^n}{n!}\right]=\frac{a_0}{n!} z^n+\frac{a_1}{(n-1)!} z^{n-1}+\cdots+a_n=\frac{1}{n!}\sum_{\nu=0}^{n} a_\nu D^\nu (z^n) \qquad \left(D\equiv\frac{d}{dz}\right),$$

$$\vdots$$

where a_i are constants which can be equal to zero.

If $f(z)=\sum_{i=0}^{n} c_i z^i$, we have

$$L(f)=\sum_{i=0}^{n} c_i L[z^i]=\sum_{i=0}^{n} c_i \sum_{k=0}^{i} a_k D^k (z^i)$$

$$=\sum_{k=0}^{n} a_k D^k\left(\sum_{i=0}^{n} c_i z^i\right)=\sum_{k=0}^{n} a_k f^{(k)}(z).$$

This proves the proposition.

2. Let the operator $L\in E$ leave invariant the bounded set K of points in the complex plane, i. e. if $f\in S$ and all the zeros of f are in K, then either $L(f)$ is identically zero or all its zeros are also in K. We shall prove that L is of the form

$$L[f(z)]=cf^{(s)}(z). \tag{4}$$

Assume the contrary: at least two coefficients in (2) are non-zero. For sufficiently large n the polynomial $L[(z-a)^n]=g(z)$ $(a\in K)$ has the zero $\alpha\neq a$. The polynomial $L[(z-b)^n]=g(z+a-b)$ $(b\in K)$ has the zero $b+(\alpha-a)$. Since b is arbitrary we have $K+(\alpha-a)\subset K$. But this is impossible for K is bounded. Hence, we have proved the following theorem.

Theorem. — *If the operator $L\in E$ leaves invariant the bounded set K of the complex plane then it has the form* (4).

REFERENCE

[1] I. Raitchinov: *Sur un théorème de Pólya.* Publications de l'Institut Mathématique de Belgrade, nouvelle série, t. 2 (16), 1962, p. 141—144.