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SMALL FORCED DAMPING VIBRATIONS OF HOMOGENEOUS TORSIONAL SYSTEM WITH SPECIAL STATIC CONSTRAINTS[1]

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1. Background. — Small vibrations of a torsional system, doubly coupled statically, considering also the influence of external and internal friction, are analysed. The system considered consists of a set flywheels attached to a light shaft with a spring and a dashpot between each flywheel and the ground (Fig. 1). It is assumed that the damping force of external and internal friction is proportional to the first power of the velocity. The kinetic and

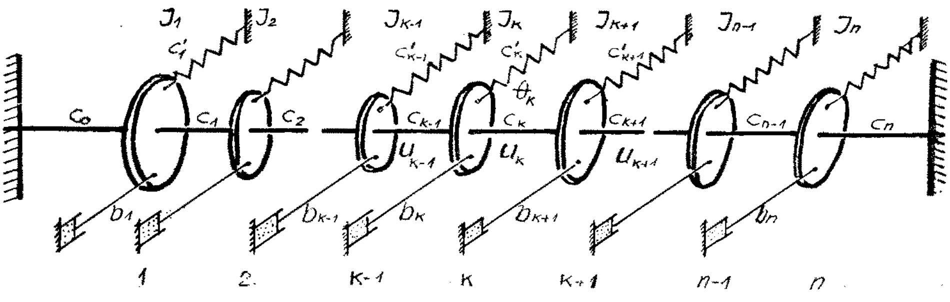

Fig. 1.

potential energies and the dissipation function of a system which performs small vibrations are homogeneous quadratic forms in the generalized angular velocities ($\dot{q}=\dot{\theta}$) and generalized coordinates ($q=\theta$) respectively, with constant coefficients, [1], namely

$$(1) \qquad 2E_k=(\dot{\theta})\, A\, \{\dot{\theta}\}, \quad 2E_p=(\theta)\,(C+C')\,\{\theta\}, \quad 2\Phi=(\dot{\theta})\,(B+U)\,\{\dot{\theta}\}.$$

Here $A\ (a_{ik})$, $a_{ii}=J_k$, $a_{ik}=0$ for $i \neq k$, represents the diagonal inertia matrix; $C\ (c_{ik})$, with $c_{ii}=c_{i-1}+c_i$, $c_{i,\, i-1}=-c_{i-1}$, $c_{i,\, i+1}=-c_i$, $c_{ik}=0$ for $|i-k|>1$ the torsional shaft rigidity matrix which has Jacobi's form with three diagonal rows, [2]. Further $C'\ (c'_{ik})$, $c'_{ii}=c'_i$, $c'_{ik}=0$ is the diagonal stiffness matrix of the spring stiffness reduced to the equivalent torsional rigidity; $B\,(b_{ii}=b_i,\ b_{ik}=0)$

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is the diagonal dissipative matrix of external friction and $U\,(u_{ik})$ the dissipative matrix of internal friction which has the same form as the matrix C changing only the letter c in the letter u; $\{\dot{\theta}\}$, $\{\theta\}$ are the column matrices and $(\dot{\theta})$, (θ) the row matrices, [3].

By Lagrange's equations, the system of differential equations of motion may be written in the matrix form

$$A\,\{\ddot{\theta}\}+(B+U)\,\{\dot{\theta}\}+(C+C')\,\{\theta\}=\{Q\} \tag{2}$$

where Q is the generalized force, the force or the moment. In this case of torsional vibrations it is the torsional moment, which is $\mathfrak{M}=0$ or $\mathfrak{M}\neq 0$ whot corresponds to free or to forced vibrations.

2. Free vibrations of a homogeneous system. — In the case of a homogeneous system, in which case the coefficients of the matrices are

$$J_k=J;\quad c_k=c;\quad c_k'=c';\quad b_k=b;\quad u_k=u;\quad c_0\neq c;\quad c_n\neq c$$

using the ratios

$$p=c/J;\ q=c'/J;\ b/J=2r;\ u/J=2\rho;\ c_0/c=\nu_0;\ c_n/c=\nu_n;\ u_0/u=\mu_0,\ u_n/u=\mu_n$$

or

$$A=JI;\quad B=bI;\quad C'=c'I;\quad C=c\,J_1;\quad U=u\,J_2,$$

the above system of differential equations (2) becomes

$$I\,\{\ddot{\theta}\}+2\,(r\,I+\rho\,J_2)\,\{\dot{\theta}\}+(q\,I+p\,J_1)\,\{\theta\}=\{\mathfrak{M}/J\};\ \text{or}\ \{0\}; \tag{3}$$

where I is the unit matrix and J_1 and J_2 the special Jacobi's matrices

$$J_1=\begin{vmatrix} \nu_0+1 & -1 & 0 & \cdot & & \\ -1 & 2 & -1 & \cdot & & \\ 0 & -1 & 2 & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & 2 & -1 \\ & & & \cdot & -1 & \nu_n+1 \end{vmatrix};\quad J_2=\begin{vmatrix} \mu_0+1 & -1 & 0 & \cdot & & \\ -1 & 2 & -1 & \cdot & & \\ 0 & -1 & 2 & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & 2 & -1 \\ & & & \cdot & -1 & \mu_n+1 \end{vmatrix}. \tag{4}$$

To solve the system of differential equations (3) we assume that is $\{\theta\}=\{r\}\,(\exp\lambda t)$ where $\{r\}$ is the complex amplitude vector and $\lambda=-\delta\pm i\,\omega$, $i=\sqrt{-1}$, the eigenvalue. The problem is then reduced to one of characteristic values, [4], and the corresponding characteristic equation is

$$f\,(\lambda)=\left|(\lambda^2+2\,r\,\lambda+q)\ \boldsymbol{I}+(2\,\rho\ \boldsymbol{J}_2\lambda+p\ \boldsymbol{J}_1)\right|=0. \tag{5}$$

Jacobi's matrices J_1 and J_2 depend on the boundary conditions, namely on the coefficients ν_0, ν_n, μ_0, μ_n. Further the three characteristic cases of the boundary conditions are treated.

a) Shaft with built -in ends. — In this case (Fig. 2 *a*) the coefficients are $c_0=c_n=c$; $\nu_0=\nu_n=1$, $\mu_0=\mu_n=1$. The both Jacobi's matrices are equal with the form

$$J_1=J_2=J=\begin{vmatrix} 2 & -1 & 0 & \cdot & & \\ -1 & 2 & -1 & \cdot & & \\ 0 & -1 & 2 & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & 2 & -1 \\ & & & \cdot & -1 & 2 \end{vmatrix} \tag{6}$$

and the corresponding characteristic polynomial is

(7) $$f(z) = |zI + J| = \sum_{\nu=0}^{n} a_\nu z^{n-\nu} = 0$$

with the eigenvalue

(8) $$z = \frac{\lambda^2 + 2r\lambda + q}{2\rho\lambda + p}.$$

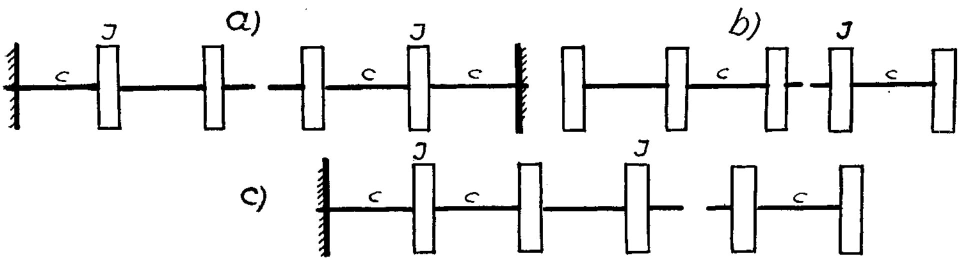

Fig. 2.

From Eq. (6) it follows that the characteristic equations can be found by the use of formulae of regression obtained by simple expansion of the determinants

(9) $$f_n(z) = (z+2)\, f_{n-1}(z) - f_{n-2}(z) = 0; \quad f_0 = 1.$$

The coefficients a_ν of the characteristic polynomial $f(z) = 0$ are $a_\nu = S_\nu$, where S_ν is the scalar of the ν-th order of the Jacobi's matrix. They can be calculated immediately or by means of the connected relation as follows

(10) $$a_\nu = S_\nu = \sum_{s=0}^{\nu} \binom{2n-\nu-s}{\nu-s}; \qquad a_\nu^{(n)} = a_\nu^{(n-1)} + 2a_{\nu-1}^{(n-1)} - a_{\nu-2}^{(n-2)},$$

and are

$$a_0 = 1; \; a_1 = 2n; \; a_2 = n(2n-3) + 1; \; \ldots; \; a_n = |J| = n + 1.$$

These coefficients are given in the Table 1. They form a series of numbers whose differences are determined $\Delta^\nu = 2^\nu$, $\Delta^{\nu+1} = 0$; ν is the order of the difference, [5].

TABLE I

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
1	1	2									
2	1	4	3								
3	1	6	10	4							
4	1	8	21	20	5						
5	1	10	36	56	35	6					
6	1	12	55	120	126	56	7				
7	1	14	78	220	330	252	84	8			
8	1	16	105	364	715	792	462	120	9		
9	1	18	136	560	1365	2002	1716	792	165	10	
10	1	20	171	816	2380	4368	4290	3432	1287	220	11
Δ^ν	0	2	4	8	16	32	64	128			

If z_s is the root of the characteristic equation (7) it follows that the eigenvalue (8) is

$$\lambda^2+2\,(r-\rho\, z_s)\,\lambda+(q-pz_s)=0; \quad \lambda_s=-\delta\pm i\,\omega; \tag{11}$$
$$\delta=r-\rho\, z_s; \quad \omega^2=(q-pz_s)-(r-\rho\, z_s)^2.$$

b) Shaft with one end built -in, other end free. — In this case (Fig. 2 *b*) the coefficients are

$$c_0=c;\ c_n=0;\ \nu_0=1;\ \nu_n=0;\ \mu_0=1;\ \mu_n=0;\ J_1=J_2,$$

and from Eq. (7) and Eq. (9) it follows that the formula of regression is

$$f_n^{(b)}(z)=(z+1)\,f_{n-1}^{(a)}(z)-f_{n-2}^{(a)}(z)=0 \tag{12}$$

where $f_n^{(a)}(z)=f_n^{(a)}$ represents the characteristic equation for the first case of the boundary conditions (Fig. 2*a*) of a system with n disks.

c) Shaft with free ends. — In this case (Fig. 2 *c*) the coefficients are

$$c_0=c_n=0;\ \nu_0=\nu_n=0;\ \mu_0=\mu_n=0$$

and it follows that the reccurence formula is

$$f_n^{(c)}=(z+1)\,f_{n-1}^{(b)}-f_{n-2}^{(b)}=(z+1)\,[(z+1)\,f_{n-2}^{(a)}-f_{n-3}^{(a)}]-$$
$$-[(z+1)\,f_{n-3}^{(a)}-f_{n-4}^{(a)}]=(z+1)\,[f_{n-1}^{(a)}-f_{n-2}^{(a)}]-[f_{n-2}^{(a)}-f_{n-3}^{(a)}]=$$
$$=f_n^{(a)}-2f_{n-1}^{(a)}+f_{n-2}^{(a)}=zf_{n-1}^{(a)}=0;\ z\neq 0$$

or

$$f_n^{(c)}(z)=f_{n-1}^{(a)}(z)=0. \tag{13}$$

Hence we can conclude that the first case is the basic one because of the possibility of becoming the characteristic equations for two other cases by means of their characteristic equations.

3. Application of trigonometric method. — The homogeneous system consists of many identical elements, hence the method of linear difference equations can be used, [6]. Let us consider a homogeneous system with boundary conditions governed by means of differential equation (3), then the problem is reduced to the system of n homogeneous linear equations with constant coefficients

$$\big((\lambda^2+2\,r\,\lambda+q)\,I+2\,\rho\lambda\,J_2+p\,J_1\big)\,\{r\}=\{0\} \tag{14}$$

or

$$\begin{array}{ll} f_0\,A_1-f\,A_2 & =0 \\ \cdots\cdots\cdots\cdots & \\ -A_{k-1}+(z+2)\,A_k-A_{k+1} & =0 \\ \cdots\cdots\cdots & \\ -f\,A_{n-1}+f_n\,A_n & =0 \end{array} \tag{15}$$

where

$$f_0=\lambda^2+2\,r\,\lambda+q+2\,\rho\,(1+\mu_0)\,\lambda+p\,(1+\nu_0);\quad f=2\,\rho\,\lambda+p;$$
$$f_n=\lambda^2+2\,r\,\lambda+q+2\,\rho\,(1+\mu_n)\,\lambda+p\,(1+\nu_n). \tag{16}$$

Introducing an auxiliary complex angle

(17) $$z+2=2\cos\varphi;\ z=-2+2\cos\varphi=-4\sin^2\frac{\varphi}{2}=\frac{\lambda^2+2r\lambda+q}{2\rho\lambda+p}$$

the difference equation for a purely homogeneous system is

(18) $$A_{k-1}-2\cos\varphi\ A_k+A_{k+1}=0$$

with corresponding solution

(19) $$A_k=A\cos k\varphi+B\sin k\varphi$$

where A and B are the constants which must satisfy the boundary conditions

(20) $$\begin{gathered} f_0 A_1-f A_2=0;\quad A_1=A\cos\varphi+B\sin\varphi;\quad A_2=A\cos 2\varphi+B\sin 2\varphi;\\ -f A_{n-1}+f_n A_n=0;\quad A_{n-1}=A\cos(n-1)\varphi+B\sin(n-1)\varphi;\\ A_n=A\cos n\varphi+B\sin n\varphi. \end{gathered}$$

These two linear homogeneous equations have the solutions for A and B different from zero if their determinant is equal to zero

$$\Delta(\varphi)=\begin{vmatrix} f_0\cos\varphi-f\cos 2\varphi & f_0\sin\varphi-f\sin 2\varphi\\ -f\cos(n-1)\varphi+f_n\cos n\varphi & -f\sin(n-1)\varphi+f_n\sin n\varphi \end{vmatrix}=0.$$

This determinant represents the characteristic equation

(21) $$\Delta(\varphi)=-f_0 f_n\sin(n-1)\varphi+f(f_0+f_n)\sin(n-2)\varphi-f^2\sin(n-3)\varphi=0.$$

For three characteristic cases of boundary conditions (Fig. 2) the results are given in the Table 2.

TABLE 2

Case	$\Delta(\varphi)=0$	φ_s	s
Fig. 2a	$\sin(n+1)\varphi$	$s\pi/(n+1)$	$1, 2, \dots, n$
Fig. 2b	$\sin(n-1)\varphi-\sin\varphi$	$(2s-1)\pi/(2n+1)$	$1, 2, \dots, n$
Fig. 2c	$\sin n\varphi$	$s\pi/n$	$1, 2, \dots, n-1$

The roots of these trigonometric equations can be determined graphically, [7].

From Eq. (17) it follows that the eigenvalue λ_s is

(22) $$\lambda_s^2+2\left(r-4\rho\sin^2\frac{\varphi}{2}\right)\lambda_s+\left(q-4p\sin^2\frac{\varphi}{2}\right)=0;\quad \lambda=-\delta\pm i\omega;$$

or

(23) $$\delta_s=r-4\rho\sin^2\frac{\varphi}{2};\ \omega_s^2=\left(q-4p\sin^2\frac{\varphi_s}{2}\right)-\left(r+4\rho\sin^2\frac{\varphi_s}{2}\right)^2$$

hence the corresponding amplitude is

(24) $$A_k^{(s)}=A^{(s)}\cos k\varphi_s+B^{(s)}\sin k\varphi_s.$$

4. Forced vibrations. — We now take the problem of forced vibrations with damping assuming that the first disk only is subjected to a periodic impressed torsional moment $\mathfrak{M}_1 = \mathfrak{M}_{10}$ (exp $i\,\Omega\, t$), with the amplitude $\mathfrak{M}_{10}$ and angular frequency Ω, and that the others are not pertubated. Supposing the stationary state of motion it is convenient to try a solution of the form $\{\theta\} = \{C\}$ (exp $i\,\Omega\, t$) where $\{C\}$ is the complex amplitude vector of the forced vibrations. The corresponding generalized force is the moment $Q = \mathfrak{M}_1$, hence Eq. (3) for a homogeneous system becomes

$$I\,\{\ddot{\theta}\} + 2\,(r\,I + \rho\,J_2)\,\{\dot{\theta}\} + (q\,I + p\,J_1)\,\{\theta\} = \{\mathfrak{M}_1/J\} \tag{25}$$

and the set of algebraic linear nonhomogeneous equations is

$$\begin{aligned}(26)\qquad (z+1+a)\,C_1 - C_2 \qquad\qquad &= \mathfrak{M}_{10}/J\,(p + 2\,\rho\, i) = h\\ \cdots\cdots\cdots\cdots\cdots\cdots &\\ -C_{k-1} + (z+2)\,C_k - C_{k+1} \quad &= 0\\ \cdots\cdots\cdots\cdots &\\ -C_{n-1} + (z+1+b)\,C_n &= 0\end{aligned}$$

where the abbreviations are

$$a = \frac{2\,\rho\, i\,\Omega\,\nu_0 + p\,\mu_0}{2\,\rho\, i\,\Omega + p}, \quad b = \frac{2\,\rho\, i\,\Omega\,\nu_n + p\,\mu_n}{2\,\rho\, i\,\Omega + p}; \quad i = \sqrt{-1}.$$

The determinant of this system of homogeneous linear equations can be determined by making use of the method of finite differences, namely by introducing an auxiliary complex angle

$$z + 2 = 2\cos\psi, \quad z + 1 = -1 + 2\cos\psi \tag{27}$$

supposing the solution in the form

$$C_k = A\cos k\,\psi + B\sin k\,\psi \tag{28}$$

where A and B are unknown complex amplitudes. Substituting Eq. (28) into Eq. (26) it follows

$$\begin{aligned}(29)\qquad &(f_0\cos\psi - \cos 2\,\psi)\,A + (f_0\sin\psi - \sin 2\,\psi)\,B = h\\ &[-\cos(n-1)\,\psi + f_n\cos n\,\psi]\,A + [-\sin(n-1)\,\psi + f_n\sin n\,\psi]\,B = 0\end{aligned}$$

with

$$f_0 = a - 1 + 2\cos\psi, \; f_n = b - 1 + 2\cos\psi = f_0 - a + b,$$

and the determinant of this system is

$$\Delta(\psi) = f_0\, f_n \sin(n-1)\,\psi - (f_0 + f_n)\sin(n-2)\,\psi + \sin(n-3)\,\psi. \tag{30}$$

By means of Cramer's rule the constants A and B and amplitude C_k are determined as follows

$$A = h\,[f_n\sin n\,\psi - \sin(n-1)\,\psi]/\Delta; \; B = -h\,[f_n\cos n\,\psi - \cos(n-1)\,\psi]/\Delta, \tag{31}$$

$$C_k = h\,[f_n\sin(n-k)\,\psi - \sin(n-1-k)\,\psi].$$

The results for three characteristic cases of boundary conditions are given in Table 3.

Let us now consider the forced vibrations of a homogeneous system with initial conditions, supposing that in the initial moment ($t = 0$) the system is at rest ($\theta = 0$, $\theta_k = 0$) and that suddenly perturbing moment depending on time (t) is applied to the last disk of the homogeneous system shown in

Fig. $2a$. In this case the system of governed differential equations is such as in Eq. (3) only with the right side $h_n(t) = \mathfrak{M}_n/J$, $\mathfrak{M}_n = \mathfrak{M}_n(t)$.

TABLE 3

Case	f_0	f_n	$\Delta(\psi)$	A	B
Fig. $2a$	$2\cos\psi$	$2\cos\psi$	$\sin(n+1)\psi$	h	$-h\,\mathrm{ctg}\,(n+1)\psi$
Fig. $2b$	$2\cos\psi$	$2\cos\psi-1$	$2\cos\frac{2n+1}{2}\psi.\ \sin\frac{\psi}{2}$	h	$h\,\mathrm{tg}\,\frac{2n+1}{2}\psi$
Fig. $2c$	$2\cos\psi-1$	$2\cos\psi-1$	$-2\sin n\psi(1-\cos\psi)$	$\frac{h[\sin(n+1)\psi-\sin n\psi]}{\Delta}$	$\frac{-h[\cos(n+1)\psi-\cos n\psi}{\Delta}$

Making use of the Laplace transform $\eta_k(\lambda)$ for the angular amplitude $\theta_k(t)$ and for perturbing moment [8]

$$\eta_k(\lambda) = L(\theta_k) = \int_0^\infty e^{-\lambda t}\,\theta_k\,dt; \quad F_n(\lambda) = L(h_n) \tag{32}$$

by means of the operational rules for derivatives

$$\int_0^\infty e^{-\lambda t}\,\dot\theta\,dt = -\theta(0) + \lambda\eta = \lambda\eta; \quad \int_0^\infty e^{-\lambda t}\,\ddot\theta\,dt = -[\lambda\theta(0) + \dot\theta(0)] + \lambda^2\eta = \lambda^2\eta$$

one obtains the following system of linear equations

$$\big((\lambda^2 + 2r\lambda + q)\,I + (2\rho\lambda + p)\,J\big)\,\{\eta\} = \{F_n\}; \ J_1 = J_2 = J \tag{33}$$

or

$$\begin{aligned} &(z+2)\,\eta_1 - \eta_2 = 0\\ &\ldots\ldots\ldots\ldots\ldots\\ &-\eta_{k-1} + (z+2)\,\eta_k - \eta_{k+1} = 0\\ &\ldots\ldots\ldots\ldots\\ &-\eta_{n-1} + (z+2)\,\eta_n = F_n/(2\rho\lambda + p). \end{aligned} \tag{34}$$

Introducing the relation (17) supposing the solution in form (19) taking into consideration the first and the last equation of the above mentioned system (34) it follows

$$A\,(2\cos\varphi - \cos 2\varphi) + B\,(\sin 2\varphi - \sin 2\varphi) = 0,$$

$$A\,[2\cos\varphi\cos n\varphi - \cos(n-1)\,\varphi] + B\,[2\cos\varphi\sin n\varphi - \sin(n-1)\,\varphi = F_n/(2\rho\lambda + p)$$

and the constants are

$$A = 0; \qquad B = \frac{F_n}{(2\rho\lambda + p)\sin(n+1)\varphi}.$$

Hence the corresponding Laplace transform is

$$\eta_k = B\sin k\varphi = \frac{F_n}{2\rho\lambda + p}\cdot\frac{\sin k\varphi}{\sin(n+1)\varphi} = F_n\frac{M(\lambda)}{N(\lambda)} \tag{35}$$

where M and N are the polynomials of λ or of z because it is $\cos\varphi = 1 + (z/2)$. They have the forms

$$M = (2\rho\lambda + p)^{-1}\sin k\varphi = (2\rho\lambda + p)^{-1}\sum_1 (-1)^{\nu-1}\binom{k}{2\nu-1}\sin^{2\nu-1}\varphi\,\cos^{k+1-2\nu}\varphi,$$

$$N = \sin(n+1)\,\varphi = \sum_1 (-1)^{\nu-1}\binom{n+1}{2\nu-1}\sin^{2\nu-1}\varphi\,\cos^{n+1-2\nu}\varphi.$$

The degree of the polynomial $M(\lambda)$ is lower than the degree of the polynomial $N(\lambda)$. The polynomial $N(\lambda)$ has all roots distinct $\varphi_s = s\pi/(n+1)$, hence the expansion method of operational calculus can be used. The quotient of polynomials M and N can be expr asessed an expansion in partial fractions

$$(36) \qquad \frac{M}{N} = \sum_{s=1}^{n} \frac{B_s}{\lambda - \lambda_s}; \quad B_s = \frac{M(\lambda_s)}{N'(\lambda_s)}; \quad N' = \frac{dN}{d\lambda} = \frac{dN}{d\varphi} \cdot \frac{d\varphi}{d\lambda}$$

and the coefficients are

$$(37) \qquad B_s = -\frac{\sin k\varphi \sin \varphi}{(n+1)\cos(n+1)\varphi} \cdot \frac{2\rho\lambda + p}{\rho\lambda^2 + p\lambda + rp - \rho q}; \quad \lambda = \lambda_s;$$

hence the Laplace transform is

$$(38) \qquad \eta_k = \sum_{s=1}^{n} B_s \frac{F_n}{\lambda - \lambda_s} = \sum_s B_s L_1 L_2.$$

By using Heaviside's expansion theorem and the theorem of the convolution, [9], we can write the amplitude $\theta_k(t)$ corresponding to the Laplace transform $\eta_k(\lambda)$ in the following form

$$(39) \qquad \theta_k(t) = -\sum_{s=1}^{n} \frac{\sin k\varphi_s \sin \varphi_s}{(n+1)\cos(n+1)\varphi_s} \cdot \frac{2\rho\lambda_s + p}{\rho\lambda_s^2 + p\lambda_s + rp - \rho q} \int_0^t e^{\lambda_s(t-\tau)} \frac{\mathfrak{M}_n(\tau)}{J} d\tau.$$

Thus in the same way we can also obtain the amplitudes for two other cases of boundary conditions (Fig. $2b$, c)

$$(40) \qquad \begin{aligned} &(\text{Fig. } 2b) \quad \eta_k = \frac{F_n}{2\rho\lambda + p} \cdot \frac{\sin k\varphi}{\sin(n+1)\varphi - \sin n\varphi}; \quad \varphi_s = \frac{2s-1}{2n+1}\pi; \\ &(\text{Fig } 2c) \quad \eta_k = \frac{F_n}{2\rho\lambda + p} \cdot \frac{\sin k\varphi}{\sin n\varphi}; \quad \varphi_s = \frac{s\pi}{n}; \end{aligned}$$

with

$$b) \qquad \frac{dN}{d\lambda} = \frac{-[(n+1)\cos(n+1)\varphi - n\cos k\varphi]}{\sin\varphi} \cdot \frac{(\rho\lambda^2 + p\lambda + rp - \rho q)}{(2\rho\lambda + p)^2};$$

$$c) \qquad \frac{dN}{d\lambda} = \frac{-n\cos n\varphi}{\sin\varphi} \cdot \frac{(\rho\lambda^2 + p\lambda + rp - \rho q)}{(2\rho\lambda + p)^2}.$$

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