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A THEOREM ON SEMIGROUPS OF LINEAR OPERATORS

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Let G_+ be a semigroup of rational numbers r of the form $l/2^k$, l, $k=0, 1, 2, \ldots$ with addition as a binary operation. Let R be n-dimensional vector space over the field of complex numbers and let $r \to \Phi(r)$ be a representation of G_+ in the semigroup of all linear transformation of R with multiplication as a binary operation. We have

$$\Phi(r)\,\Phi(r') = \Phi(r+r') \qquad (r, r' \in G_+). \tag{1}$$

We shall say that $\{\Phi(r)\}$ is a regular representation of G_+ if $\Phi(r)$ is regular for every $r \in G_+$. We shall denote by $\overline{\Phi}(r)$ the matrix of $\Phi(r)$ in a fixed basis of R.

S. Kurepa [1] has proved that if $\{\Phi(r)\}$ is a regular representation of G_+ then in a suitable basis of R we have

$$\Phi(r) = U(r)\ \exp\ (r\,C) = \exp\,(r\,C)\ U(r) \tag{2}$$

where $\overline{U}(r)$ is a semigroup of diagonal unitary matrices and $\overline{C}$ is the sum of a nilpotent and a real diagonal matrix.

We shall obtain Kurepa's result in a simpler and quite different way.

By $[A, B]$ we denote the commutator of A and B. We have

Theorem 1. *If $\{\Phi(r)\}$ is a regular representation of G_+ then in suitable basis $\Phi(r) = U(r)\exp(r\,C)$ where $[C, U(r)] = 0$ for every $r \in G_+$ and $\overline{U}(r)$ is a semigroup of diagonal unitary matrices.*

Proof. Since $\Phi(r)$ is regular we can write $\Phi(r) = \exp\,\Psi(r)$ where $\Psi(r)$ is a polynomial in $\Phi(r)$, (see [2]). From (1) we find that $[\Phi(r), \Phi(r')] = 0$ so that also $[\Psi(r), \Psi(r')] = 0$. Hence,

$$\exp\,[\Psi(r) + \Psi(r')] = \exp\ \Psi(r+r'),$$

$$\Psi(r) + \Psi(r') = \Psi(r+r') + 2\,\pi\,i\,D(r, r'), \tag{3}$$

where, in suitable basis, $\overline{D}(r, r')$ is a diagonal matrix with real integral elements. If $\psi_{ij}(r)$ is the element in the i-th row and j-th column of $\overline{\Psi}(r)$ we find from (3) that

$$\psi_{ij}(r) + \psi_{ij}(r') = \psi_{ij}(r+r') \qquad (i \neq j). \tag{4}$$

The general solution of this equation is given by $\psi_{ij}(r) = r\,\alpha_{ij}$ where α_{ij} is a complex constant. The real part of $\psi_{ii}(r)$ also satisfies the functional equation (4) and $Re\ \psi_{ii}(r) = r\,\alpha_{ii}$. Hence, we have

$$\Psi(r) = r\,A + i\,\Psi_1(r) \tag{5}$$

where $\bar{A}=\|\alpha_{ij}\|$ and $\bar{\Psi}_1(r)$ is a real diagonal matrix such that

$$\Psi_1(r)+\Psi_1(r')=\Psi_1(r+r')+2\pi D(r, r'). \tag{6}$$

From $[\Psi(r), \Psi(r')]=0$ and (5) we get $[A, r'\Psi_1(r)-r\Psi_1(r')]=0$. Putting $r'=1$ we find $[A, \Psi_1(r)-r\Psi_1(1)]=0 \Rightarrow [A+i\Psi_1(1), \Psi_1(r)-r\Psi_1(1)]=0$.

Finally, we have

$$\begin{aligned}\Phi(r) &= \exp\Psi(r)= \exp\{r[A+i\Psi_1(1)]+i[\Psi_1(r)-r\Psi_1(1)]\}\\ &= \exp i[\Psi_1(r)-r\Psi_1(1)]\exp r[A+i\Psi_1(1)]\\ &= U(r)\exp(rC)\end{aligned}$$

where $U(r)=\exp i[\Psi_1(r)-r\Psi_1(1)]$, $C=A+i\Psi_1(1)$. Using (6) we can check that $\bar{U}(r)$ is a semigroup of diagonal unitary matrices. Since $U(r)$ is a polynomial in $\Psi_1(r)-r\Psi_1(1)$ it commutes with C.

Thus theorem 1 is completely proved.

Corollary 1. *Under the conditions of theorem* 1 *there is a basis in* R *such that in this basis we have*

$$\bar{\bar{\Phi}}(r)=\bar{\bar{V}}(r)\exp(r\bar{\bar{J}}), \qquad [\bar{\bar{V}}(r), \bar{\bar{J}}]=0 \tag{7}$$

where $\{\bar{\bar{V}}(r)\}$ *is a semigroup of diagonal unitary matrices and* $\bar{\bar{J}}$ *is a Jordan (i.e. classical canonical) matrix with real eigenvalues.*

Proof. Since $\bar{U}(r)$ is diagonal and commutes with $\bar{C}$ for every $r\in G_+$ we conclude that if some entry c_{ij} of $\bar{C}$ is non-zero then in every $\bar{U}(r)$ i-th and j-th diagonal elements are equal (see [2]). Consequently, we can find a permutation matrix $\bar{P}$ such that

$$\bar{P}^{-1}\bar{C}\bar{P}=\bar{M}_1\dot{+}\bar{M}_2\dot{+}\cdots\dot{+}\bar{M}_k=\bar{M}$$

$$\bar{P}^{-1}\bar{U}(r)\bar{P}=\bar{N}_1(r)\dot{+}\bar{N}_2(r)\dot{+}\cdots\dot{+}\bar{N}_k(r)=\bar{N}(r) \qquad (r\in G_+)$$

where $\bar{N}_i(r)$ is scalar unitary matrix of order n_i and $\bar{M}_i$ is of the same order. We have $\bar{P}^{-1}\bar{\Phi}(r)\bar{P}=\bar{N}(r)\exp(r\bar{M})$.

Let $\bar{T}=\bar{T}_1\dot{+}\bar{T}_2\dot{+}\cdots\dot{+}\bar{T}_k$ be a quasidiagonal matrix of the same type as $\bar{M}$ such that $\bar{K}=\bar{T}^{-1}\bar{M}\bar{T}$ is Jordan matrix. Then $\bar{T}^{-1}\bar{N}(r)\bar{T}=\bar{N}(r)$ since $\bar{N}_i(r)$ are scalar matrices, and $(\bar{P}\bar{T})^{-1}\bar{\Phi}(r)(\bar{P}\bar{T})=\bar{N}(r)\exp(r\bar{K})$. Let $\bar{K}=\bar{K}_1+ +i\bar{K}_2$ where $\bar{K}_1$ and $\bar{K}_2$ are real matrices. Since $[\bar{K}_1, \bar{K}_2]=0$ we obtain $(\bar{P}\bar{T})^{-1}\bar{\Phi}(r)(\bar{P}\bar{T})=\bar{N}(r)\exp(ir\bar{K}_2)\exp(r\bar{K}_1)=\bar{\bar{V}}(r)\exp(r\bar{\bar{J}})$ with $\bar{\bar{V}}(r)=\bar{N}(r)\exp(ir\bar{K}_2)$ and $\bar{\bar{J}}=\bar{K}_1$. In the new basis determined by the matrix $(\bar{P}\bar{T})$ we have (7).

Corollary 2. If $\{\Phi(r)\}$ is a regular representation of G_+ and $\Phi(r_0)=E$ (= identity operator) for some $r_0\neq 0$ $(r_0\in G_+)$ then in a suitable basis $\{\bar{\Phi}(r)\}$ is a semigroup of diagonal unitary matrices.

Proof. Using corollary 1 we find that $\bar{\bar{V}}(r_0)\exp(r_0\bar{\bar{J}})=\bar{E}\Rightarrow\exp(r_0\bar{\bar{J}})= =\bar{\bar{V}}(r_0)^{-1}$. But the last equality holds if and only if $\bar{\bar{J}}=0$ so that $\bar{\Phi}(r)=\bar{\bar{V}}(r)$.

REFERENCES

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