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SLOWLY OSCILLATING FUNCTIONS AND THEIR APPLICATION TO THE ASYMPTOTIC EVALUATIONS OF THE RIESZ AND $G_\theta^\varkappa$ MEANS OF MULTIPLE FOURIER SERIES

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1. Let $Q=(x_1, x_2, \ldots, x_n)$ be a point in the n-dimensional Euclidean space and $f(Q)\equiv f(x_1, x_2, \ldots, x_n)$ a real-valued, L-integrable function having the period 2π in each variable. Let

$$f(Q)\sim \sum_{-\infty}^{+\infty}\cdots \sum a_{m_1\ldots m_n} \exp\Big(i\sum_{j=1}^{n} m_j x_j\Big) \tag{1}$$

be its Fourier series where

$$a_{m_1\ldots m_n}=(2\pi)^{-n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f(Q)\exp\Big(-i\sum_{j=1}^{n} m_j x_j\Big)\,dx_1\ldots dx_n.$$

Let us denote

$$A_k(Q)=\sum_{m_1^2+\cdots+m_n^2=k} a_{m_1\ldots m_n}\exp\Big(i\sum_{j=1}^{n} m_j x_j\Big) \tag{2}$$

and $A_k(Q)\equiv 0$ if k cannot be represented as a sum of n squares. Further we denote the spherical partial sum of order k of the series (1) by

$$S_k(Q)=\sum_{r=0}^{k} A_r(Q). \tag{3}$$

The Riesz mean of the spherical partial sums (3) is defined by [4]

$$\begin{aligned} S_x^\delta(Q) &= \sum_{m_1^2+\cdots+m_n^2\leqslant x}\{1-(m_1^2+\cdots+m_n^2)\,x^{-2}\}^\delta\, a_{m_1\ldots m_n}\exp\Big(i\sum_{j=1}^{m} m_j x_j\Big)\\ &=\sum_{r=0}^{k}(1-rx^{-2})^\delta A_r(Q),\qquad k\leqslant x^2<k+1\\ &=2\,\delta\, x^{-2}\int_0^x (1-t^2x^{-2})^{\delta-1}\, t\,S(t)\,dt, \end{aligned} \tag{4}$$

where $S(x)\equiv S_x^0(Q)\equiv S_k(Q)$.

Let $f_P(t)$ be the spherical average of the function $f(Q)$ over a sphere whose radius is t and whose centre is at the fixed point P, i. e.

$$f_P(t) = 2^{-1}\pi^{-n/2}\Gamma(n/2)\int_F f(P+\xi t)\,d\sigma_\xi \tag{5}$$

where

$$P+\xi t = (x_1^0+\xi_1 t, \ldots, x_n^0+\xi_n t),$$

F is the unit sphere $\xi_1^2+\cdots+\xi_n^2=1$, and $d\sigma_\xi$ its $(n-1)$ dimensional volume element.

There is well-known Bochner's formula [2] which expresses the Riesz mean $S_x^\delta(P)$, defined by (4), in terms of the spherical average $f_P(t)$ of $f(Q)$ at the point P. Namely, if $\delta>(n-1)/2$, then

$$S_x^\delta(P) = c x^n \int_0^\infty t^{n-1} f_P(t)\, V_{\delta+n/2}(tx)\,dt, \tag{6}$$

with

$$c = 2^{1+\delta-n/2}\Gamma(1+\delta)\{\Gamma(n/2)\}^{-1},$$

and

$$V_\mu(z) = z^{-\mu} J_\mu(z)$$

where $J_\mu(z)$ is the Bessel function of the first kind and of order μ.

Further, we need the following asymptotic relation [4]: *If* $\eta>0$ *and* $\delta>(n-1)/2$, *then*

$$x^n \left| \int_\eta^\infty t^{n-1} f_P(t)\, V_{\delta+n/2}(tx)\,dt \right| = O(x^{-\delta+(n-1)/2}) = o(1), \quad x\to\infty \tag{7}$$

uniformly in P.

We have to mention two well-known theorems [4] which connect the asymptotic behaviour of the Riesz mean $S_x^\delta(P)$ as $x\to\infty$ with the asymptotic behaviour of the spherical average $f_P(t)$ as $t\to 0$.

Theorem A. *If* $f_P(t)\to s$ *as* $t\to 0$, *for a fixed* P *or, more generally, if*

$$\lim_{t\to 0} t^{-n}\int_0^t \tau^{n-1}|f_P(\tau)-s|\,d\tau = 0,$$

then for $\delta>(n-1)/2$

$$\lim_{x\to\infty} S_x^\delta(P) = s.$$

Theorem B. *If at a fixed point* P

$$f_P(t)-s = O(t^\lambda), \quad \lambda>0, \text{ as } t\to 0,$$

then for $\delta>\lambda+(n-1)/2$

$$S_x^\delta(P)-s = O(x^{-\lambda}), \quad x\to\infty.$$

2. A function $L(x)$ defined for $x\geqslant 0$ belongs to the class of slowly oscillating functions at infinity if

a) $L(x)$ is positive and continuous in $0\leqslant x<\infty$;

b) $\lim\limits_{x\to\infty}\dfrac{L(tx)}{L(x)}=1$ for every fixed $t>0$.

This definition is due to Karamata [5, 6]. He also proved that a slowly oscillating function can be represented by

(8) $$L(x)=c(x)\exp\left\{\int_1^x t^{-1}\varepsilon(t)\,dt\right\},$$

where $c(x)$ is a positive, continuous function which tends to a positive limit as $x\to\infty$ and $\varepsilon(x)$ is a continuous function which tends to zero as $x\to\infty$.

From (8) many properties of slowly oscillating functions can be obtained. We shall mention the following [5, 6]:

(i) The asymptotic relation

$$\lim_{x\to\infty}\frac{L(tx)}{L(x)}=1$$

holds uniformly on every closed interval $a\leqslant t\leqslant b$, $(0<a<b<\infty)$.

(ii) If $L_1(x)\sim L(x)$, $x\to\infty$ then also $L_1(x)$ is a slowly oscillating function.

(iii) If $\lambda>0$, then

$$x^{\lambda}L(x)\to\infty,\quad x^{-\lambda}L(x)\to 0,\quad x\to\infty.$$

(iv) If $\lambda>0$ and

$$L_1(x)=x^{-\lambda}\max_{0\leqslant t\leqslant x}\{t^{\lambda}L(t)\},\quad L_2(x)=x^{\lambda}\max_{x\leqslant t<\infty}\{t^{-\lambda}L(t)\},$$

then $L_1(x)\cong L(x)$ and $L_2(x)\cong L(x)$, $x\to\infty$, and according to (ii) both $L_1(x)$ and $L_2(x)$ are slowly oscillating functions; $x^{\lambda}L_1(x)$ is monotonely increasing, and $x^{-\lambda}L_2(x)$ is monotonely decreasing.

(v) If $g(t)$ is such that both integrals

(9) $$\int_0^1 t^{-a}\,|g(t)|\,dt \quad\text{and}\quad \int_1^\infty t^{a}\,|g(t)|\,dt$$

exist for some $a>0$, then [1]

(10) $$\int_0^\infty g(t)\;L(tx)\;dt\cong L(x)\int_0^\infty g(t)\,dt,\;\; x\to\infty.$$

3. The slowly oscillating functions appear naturally in problems connected with asymptotic evaluations of certain integrals and sums. In [3] Bojanić considered such a problem concerning the theory of multiple Fourier series, and in the case of two-dimensional Euclidean space he proved a slightly more general theorem of the type of theorems A and B, i. e.

Theorem C. *Let* $\delta>\frac{1}{2}$ *and* $\frac{1}{2}-\delta<\alpha<2$. *If at a fixed point* $P(x_0,y_0)$

$$f_P(t)\cong t^{-\alpha}L(1/t),\quad t\to 0$$

where $L(x)$ *is a slowly oscillating function at infinity, then*

(11) $$S_x^{\delta}(P)\cong\frac{1}{2^{\alpha}}\;\frac{\Gamma(1+\delta)\,\Gamma(1-\alpha/2)}{\Gamma(1+\delta+\alpha/2)}\,x^{\alpha}L(x),\quad x\to\infty.$$

Now we are going to formulate a theorem, analogous to theorem C, in the n-dimensional Euclidean space, which connects the asymptotic behaviour of the Riesz mean of the spherical partial sums of multiple Fourier series of $f(Q)\equiv f(x_1, x_2, \ldots, x_n)$ at the point P with the asymptotic behaviour of the spherical averrage $f_P(t)$.

Theorem 1. *Let* $\delta>(n-1)/2$ *and*

$$(n-1)/2-\delta<\alpha<n. \tag{12}$$

If at a fixed point $P(x_1^0, \ldots, x_n^0)$

$$f_P(t)\cong t^{-\alpha} L(1/t, \quad t\to 0 \tag{13}$$

where $L(x)$ *is a slowly oscillating function at infinity, then*

$$S_x^\delta(P)\cong\frac{1}{2^\alpha}\,\frac{\Gamma(1+\delta)\,\Gamma\{(n-\alpha)/2\}}{\Gamma(n/2)\,\Gamma(1+\delta+\alpha/2)}\,x^\alpha L(x), \quad x\to\infty. \tag{14}$$

Proof. We give the proof of this theorem in a shorter form, for it is similar to the proof of theorem C. The theorem C is a particular case of theorem 1 when $n=2$. We make use of the Bochner's formula (6) written in the form

$$\begin{aligned} S_x^\delta(P) &= c\,x^n\int_0^\eta t^{n-1-\alpha}\,V_{\delta+n/2}(tx)\,L(1/t)\,dt\\ &\quad+c\,x^n\int_0^\eta t^{n-1}\,V_{\delta+n/2}(tx)\,\{f_P(t)-t^{-\alpha}L(1/t)\}\,dt\\ &\quad+c\,x^n\int_\eta^\infty t^{n-1}f_P(t)\,V_{\delta+n/2}(tx)\,dt\\ &=I_1+I_2+I_3 \end{aligned} \tag{15}$$

where, by assumption (13), η can be chosen so that

$$|f_P(t)-t^{-\alpha}L(1/t)|\leqslant\varepsilon\,t^{-\alpha}L(1/t) \ \text{ for } \ 0\leqslant t\leqslant\eta. \tag{16}$$

Thus

$$\begin{aligned} |I_2| &\leqslant\varepsilon\,c\,x^n\int_0^\eta t^{n-1-\alpha}\,|V_{\delta+n/2}(tx)|\,L(1/t)\,dt\\ &\leqslant\varepsilon\,c\,x^\alpha\int_0^\infty t^{\alpha-1-n}\,|V_{\delta+n/2}(1/t)|\,L(xt)\,dt \end{aligned}$$

in virtue of (16). Since

$$|V_{\delta+n/2}(u)|\leqslant M_1, \ \text{ for } \ 0\leqslant u\leqslant 1$$

and

$$|V_{\delta+n/2}(u)|\leqslant M_2\,u^{-\delta-(n+1)/2}, \ \text{ for } \ u\geqslant 1,$$

it follows by (12) that the function $t^{\alpha-1-n}\,V_{\delta+n/2}(1/t)$ satisfies the conditions (9). Therefore, according to (10)

$$\int_0^\infty t^{\alpha-1-n}\,|V_{\delta+n/2}(1/t)|\,L(xt)\,dt\cong L(x)\int_0^\infty t^{\alpha-1-n}\,|V_{\delta+n/2}(1/t)|\,dt, \quad x\to\infty,$$

whence

$$I_2 = o\{x^\alpha L(x)\}, \quad x \to \infty. \tag{17}$$

In virtue of (7) we have

$$|I_3| \leqslant \frac{M_3}{x^{\delta-(n-1)/2}} = \frac{M_3}{x^{\alpha+\delta-(n-1)/2} L(x)} x^\alpha L(x).$$

Since by (12) $\alpha+\delta-(n-1)/2>0$, we get by the property (iii) of slowly oscillating functions

$$I_3 = o\{x^\alpha L(x)\}, \quad x \to \infty. \tag{18}$$

Now we are going to evaluate the integral

$$\begin{aligned} I_1 &= c x^n \int_0^\eta t^{n-1-\alpha} V_{\delta+n/2}(tx) L(1/t)\, dt \\ &= c x^\alpha \Big(\int_0^\infty - \int_0^{1/x\eta} \Big) t^{\alpha-1-n} V_{\delta+n/2}(1/t) L(tx)\, dt \\ &= I_{11} + I_{12}. \end{aligned} \tag{19}$$

Since the function $t^{\alpha-1-n} V_{\delta+n/2}(1/t)$ satisfies the conditions (9), it follows from (10)

$$I_{11} \simeq c x^\alpha L(x) \int_0^\infty t^{n-1-\alpha} V_{\delta+n/2}(t)\, dt, \quad x \to \infty.$$

In virtue of the formula [4]

$$\int_0^\infty t^{\mu-1} V_\nu(t)\, dt = 2^{\mu-\nu-1} \Gamma(\mu/2) \{\Gamma(1+\nu-\mu/2)\}^{-1}, \quad 0<\mu<\nu+3/2$$

we have by (12)

$$\int_0^\infty t^{n-1-\alpha} V_{\delta+n/2}(t)\, dt = 2^{-\alpha-\delta-1+n/2} \Gamma\{(n-\alpha)/2\} \{\Gamma(1+\delta+\alpha/2)\}^{-1}.$$

Therefore,

$$I_{11} \simeq \frac{1}{2^\alpha} \frac{\Gamma(1+\delta)\Gamma\{(n-\alpha)/2\}}{\Gamma(n/2)\Gamma(1+\delta+\alpha/2)} x^\alpha L(x), \quad x \to \infty. \tag{20}$$

As

$$|V_{\delta+n/2}(1/t)| = O\{t^{\delta+(n+1)/2}\}, \quad t \to 0$$

it follows

$$\begin{aligned} |I_{12}| &\leqslant M_4 x^{-\delta+(n-1)/2} \int_0^{1/\eta} t^{\alpha+\delta-(n+1)/2} L(t)\, dt \\ &\leqslant \frac{M_5}{x^{\delta-(n-1)/2}} = \frac{M_5}{x^{\alpha+\delta-(n-1)/2} L(x)} x^\alpha L(x). \end{aligned}$$

In virtue of (12), and by the property (iii) of slowly oscillating functions

$$I_{12} = o\{x^\alpha L(x)\}, \quad x \to \infty. \tag{21}$$

Now from (19), (20) and (21)

$$I_1 \cong \frac{1}{2^\alpha} \frac{\Gamma(1+\delta)\,\Gamma\{(n-\delta)/2\}}{\Gamma(n/2)\,\Gamma(1+\delta+\alpha/2)}\, x^\alpha L(x), \quad x \to \infty . \tag{22}$$

Therefore from (15), (17), (18) and (22) follows (14) and the theorem 1 is proved.

4. Using theorem 1 we shall prove a theorem which connects the asymptotic behaviour of the $G_\theta^\varkappa$—mean of the spherical partial sums of multiple Fourier series of $f(Q) \equiv f(x_1, \ldots, x_n)$ at the point P with the asymptotic behaviour of the spherical average $f_P(t)$.

The method of summation $G_\theta^\varkappa$[7] is defined by

$$G_\theta^\varkappa(\lambda; x) = \sum_{\lambda_\nu \leqslant x} \{1 - \exp[(\lambda_\nu - x)\, x^{-\theta}]\}^\varkappa a_\nu$$

$$(0 \leqslant \lambda_0 < \lambda_1 < \cdots < \lambda_\nu < \cdots \to \infty, \ \nu \to \infty)$$

where

$$0 < \theta < 1 \text{ and } \varkappa > 0,$$

or by

$$G_\theta^\varkappa(x) = \int_0^x \{1 - \exp[(t-x)\, x^{-\theta}]\}^\varkappa\, d\{A(t)\},$$

where $A(t)$ is of bounded variation in every finite interval. Without loss of generality we can suppose $A(0) = 0$ and in this case the expression (23) may be written in the form

$$G_\theta^\varkappa(x) = \varkappa x^{-\theta} \int_0^x \{1 - \exp[(t-x)\, x^{-\theta}]\}^{\varkappa-1} \exp[(t-x)\, x^{-\theta}]\, A(t)\, dt.$$

The $G_\theta^\varkappa$—mean of the spherical partial sums (3) of the series (1) is defined by

$$G_\theta^\varkappa(S;\, x) = \sum_{\lambda_\nu \leqslant x} \{1 - \exp[(\lambda_\nu - x)\, x^{-\theta}]\}^\varkappa A_\nu(Q)$$

or by

$$G_\theta^\varkappa(S; x) = \varkappa x^{-\theta} \int_0^x \{1 - \exp[(t-x)\, x^{-\theta}]\}^{\varkappa-1} \exp[(t-x)\, x^{-\theta}]\, S(t)\, dt \tag{23}$$

where, without loss of generality, we have assumed that $S(0) = 0$.

Theorem 2. *Let*

$$-\frac{1}{2} < \alpha < n \textit{ and } \delta > (n-1)/2 \tag{24}$$

where δ is a positive integer.

If at a fixed point $P(x_1^0, x_2^0, \ldots, x_n^0)$

$$f_P(t) \cong t^{-\alpha} L(1/t), \quad t \to 0 \tag{25}$$

where $L(x)$ is a slowly oscillating function at infinity, then

$$G_\theta^\varkappa(S; x) = O\{x^{\alpha + \delta(1-\theta)} L(x)\}, \quad x \to \infty \tag{26}$$

for $\varkappa \geqslant \delta$.

Proof. We denote by $S^\delta(x)$ the Riesz mean of the spherical partial sums (3) of series (1). According to (4) we have

$$x^{2\delta} S^\delta(x) = 2\delta \int_0^x (x^2-t^2)^{\delta-1}\, t\, S(t)\, dt,$$

hence

(27) $$S(x) = \frac{2^{-\delta}}{S!} \frac{d^\delta}{(xdx)^\delta} \{x^{2\delta} S^\delta(x)\},$$

where the symbol $d^\delta/(xdx)^\delta$ means that the expression $x^{2\delta} S^\delta(x)$ is to be differentiated δ-times and after each differentiation the obtained result is to be divided by x, i. e.

(28) $$S(x) = \frac{2^{-\delta}}{\delta!} \frac{1}{x} \frac{d}{dx} \left\{ \frac{1}{x} \frac{d}{dx} \left[\cdots \frac{1}{x} \frac{d}{dx} (x^{2\delta} S^\delta(x)) \right] \right\}.$$

Let us apply now the $G_\theta^\varkappa$—method of summation to the function $S(x)$ defined by (27) or (28). In virtue of (23) and (27) we have

(29) $$G_\theta^\varkappa(S; x) = \frac{2^{-\delta}}{\delta!} \frac{\varkappa}{x^\theta} \int_0^x w(t, x) \frac{d^\delta}{(tdt)^\delta} \{t^{2\delta} S^\delta(t)\}\, dt,$$

where

(30) $$w = w(t, x) = \{1 - \exp[(t-x)x^{-\theta}]\}^{\varkappa-1} \exp[(t-x)x^{-\theta}].$$

Since

$$\frac{d^\delta}{(tdt)^\delta} \{t^{2\delta} S^\delta(t)\} = \sum_{\nu=0}^{\delta} p_{\delta-\nu}^{(\delta)}\, t^{-\delta+\nu} \frac{d^\nu}{dt^\nu} \{t^\delta S^\delta(t)\}$$

where $p_{\delta-\nu}^{(\delta)}$ are independent of t, and $p_0^{(\delta)} = 1$, we can write (29) in the following form

(31) $$G_\theta^\varkappa(S; x) = \frac{2^{-\delta}}{\delta!} \frac{\varkappa}{x^\theta} \sum_{\nu=0}^{\delta} p_{\delta-\nu}^{(\delta)} Q_\nu(x),$$

where

(32) $$Q_\nu(x) = \int_0^x w(t, x)\, t^{-\delta+\nu} \frac{d^\nu}{dt^\nu} \{t^\delta S^\delta(t)\}.$$

Since

$$\frac{d^\nu}{dt^\nu} \{t^\delta S^\delta(t)\} = \sum_{m=1}^{\nu+1} \frac{(-1)^{\nu+m-1} \gamma_m^{(\nu)}}{t^{\delta+\nu-2(m-1)}} \int_0^t (t^2-\tau^2)^{\delta-m}\, \tau S(\tau)\, d\tau$$

where $\gamma_m^{(\nu)} > 0$ depend on ν, but are independent of t, and $\gamma_0^{(\nu)} = 1$, it follows

(33) $$\lim_{t\to 0} \frac{1}{t^{\delta-\nu}} \frac{d^{\nu-1}}{dt^{\nu-1}} \{t^\delta S^\delta(t)\} = 0$$

for $\nu = 1, 2, \ldots, \delta$.

Further,

(34) $$\frac{d^\nu}{dt^\nu} \{w(t, x)\, t^{-\delta+\nu}\} = \sum_{s=0}^{\nu} c_{\nu-s}^{(\nu)}\, w^{(s)}(t)\, t^{-\delta+s}$$

for $\nu=0, 1, 2, \ldots, \delta$, where $c_{\nu-s}^{(\nu)}$ are independent of t, and $c_0^{(\nu)}=1$. By $w^{(s)}(t)$ is denoted the derivative of order s of the function w (30) with respect to t. Integrating by parts (32). and according to (33) and (34) the expression (31) can be written as

$$G_\theta^\varkappa(S;x)=\frac{2^{-\delta}}{\delta!}\frac{\varkappa}{x^\theta}\sum_{\nu=0}^{\delta}p_{\delta-\nu}^{(\delta)}\left\{\sum_{s=0}^{\nu}c_{\nu-s}^{(\nu)}\int_0^x w^{(s)}(t)\,t^s\,S^\delta(t)\,dt\right\},$$

or

$$G_\theta^\varkappa(S;x)=\frac{2^{-\delta}}{\delta!}\frac{\varkappa}{x^\theta}\sum_{m=0}^{\delta}q_m\int_0^x w^{(m)}(t)\,t^m\,S^\delta(t)\,dt \tag{35}$$

where q_m depend on δ, but are independent of x, and $q_\delta=1$.

In virtue of (24) the conclusion (14) of the theorem 1 holds, i. e.

$$S^\delta(x)\cong\beta x^\alpha L(x),\quad x\to\infty$$

with

$$\beta=2^{-\alpha}\,\Gamma(1+\delta)\,\Gamma\{(n-1)/2\}\,\{\Gamma(n/2)\,\Gamma(1+\delta+\alpha/2)\}^{-1},$$

and we can write

$$S^\delta(x)=\beta x^\alpha L(x)\;\lambda(x) \tag{36}$$

where

$$\lambda(x)=1+\varepsilon(x),\text{ and }\varepsilon(x)\to0\text{ when }x\to\infty. \tag{37}$$

Substituting (36) for $S^\delta(t)$ in (35) we have

$$G_\theta^\varkappa(S;x)=\frac{2^{-\delta}}{\delta!}\beta\varkappa\sum_{m=0}^{\delta}q_m F_m(x) \tag{38}$$

where

$$F_m(x)=x^{-\theta}\int_0^x w^{(m)}(t)\,t^{m+\alpha}\,L(t)\,\lambda(t)\,dt, \tag{39}$$

and [7]

$$\begin{aligned}w^{(m)}(t)=x^{-m\theta}\{1-\exp[(t-x)x^{-\theta}]\}^{\varkappa-(m+1)}\exp[(t-x)x^{-\theta}]\\ \cdot P_m\{\exp[(t-x)x^{-\theta}]\},\end{aligned} \tag{40}$$

where P_m is a polynomial of degree m in terms of $\exp[(t-x)x^{-\theta}]$. The zeros of this polynomial, with respect to $\exp[(t-x)x^{-\theta}]$ do not depend on x, but only on $\varkappa$. The asymptotic relation

$$t_\mu^{(m)}=x-x^\theta j_\mu^{(m)}(\varkappa)\cong x,\quad x\to\infty. \tag{41}$$

Let the polynomial P_m have s zeros $t_\mu^{(m)}$ $(\mu=1, 2, \ldots, s)$ in the interval $(0, x)$. Therefore the sign of $w^{(m)}(t)$ does not change within any of the intervals

$$t_\mu^{(m)}<t<t_{\mu+1}^{(m)}\quad(\mu=0, 1, 2, \ldots, s),\quad\text{where }t_0^{(m)}=0,\; t_{s+1}^{(m)}=x.$$

This property of $w^{(m)}(t)$ and the asymptotic relation (41) will be used for evaluation of the integral $F_m(x)$ which we shall write in the form

$$F_m(x)=x^{-\theta}\left(\int_0^N+\int_N^{t_1^{(m)}}+\sum_{r=2}^{s}\int_{t_{r-1}^{(m)}}^{t_r^{(m)}}+\int_{t_s^{(m)}}^{x}\right)w^{(m)}(t)\,t^{m+\alpha}\,L(t)\,\lambda(t)\,dt$$

$$(42) \qquad = H_0^{(m)}(x) + H_N^{(m)}(x) + \sum_{r=2}^{s} H_r^{(m)}(x) + H_*^{(m)}(x).$$

At first we shall evaluate the integral $H_0^{(m)}(x)$. $|\lambda(t)| \leqslant 1 + |\varepsilon(t)| \leqslant 1 + \varepsilon_0$ in the interval $(0, N)$, therefore

$$|H_0^{(m)}(x)| \leqslant (1+\varepsilon_0)\, x^{-\theta} \int_0^N |w^{(m)}(t)|\, t^{m+\alpha} L(t)\, dt.$$

Suppose that $w^{(m)}(t) > 0$ in the interval $(0, N)$. The proof is the same also if we suppose that $w^{(m)}(t) < 0$ within this interval. By the definition of slowly oscillating functions $L(t) \leqslant L_0$ for $0 \leqslant t \leqslant N$ where L_0 is a positive constant. Then according to (40) it follows

$$|H_0^{(m)}(x)| \leqslant (1+\varepsilon_0)\, L_0\, x^{-(m+1)\theta} \int_0^N \{1 - \exp[(t-x)x^{-\theta}]\}^{-\varkappa-(m+1)}$$

$$\cdot \exp[(t-x)x^{-\theta}]\, P_m\{\exp[(t-x)x^{-\theta}]\}\, t^{m+\alpha}\, dt.$$

Since we can choose x so large that for $0 \leqslant t \leqslant N$ and $m = 0, 1, 2, \ldots, \delta$

$$\{1 - \exp[(t-x)x^{-\theta}]\}^{\varkappa-(m+1)}\, P_m\{\exp[(t-x)x^{-\theta}]\} \leqslant M_0^{(m)},$$

where $M_0^{(m)}$ is a positive constant, we have

$$|H_0^{(m)}(x)| \leqslant (1+\varepsilon_0)\, L_0\, M_0^{(m)}\, x^{-(m+1)\theta} \int_0^N t^{m+\alpha} \exp[(t-x)x^{-\theta}]\, dt$$

$$\leqslant K_0^{(m)}\, x^{-(m+1)\theta} \exp[(N-x)x^{-\theta}] \int_0^N t^{m+\alpha}\, dt$$

where $K_0^{(m)} = (1+\varepsilon_0)\, L_0\, M_0^{(m)}$.

According (24) $m + \alpha > -1$, therefore

$$(43) \qquad |H_0^{(m)}(x)| \leqslant (1+\varepsilon_0)\, L_1\, M_0^{(m)}\, N^{m+\alpha+1}\, x^{-(m+1)\theta} \exp[(N-x)x^{-\theta}],$$

where $L_1 = L_0(m + \alpha + 1)$.

The integrals $H_N^{(m)}(x)$ have to be cosidered for $m = 1, 2, 3, \ldots, \delta$. Since we have supposed that $w^{(m)}(t) > 0$ in the interval $(0, N)$ and that $t_1^{(m)}$ is the first zero of the polynomial P_m, counting from the left side, then $w^{(m)}(t) > 0$ also in the interval $(N, t_1^{(m)})$. Further $|\lambda(t)| \leqslant 1 + |\varepsilon(t)| \leqslant 1 + \varepsilon_1^{(m)}(N)$,

and so

$$|H_N^{(m)}(x)| \leqslant (1 + \varepsilon_1^{(m)}(N))\, x^{-\theta} \int_N^{t_1^{(m)}} w^{(m)}(t)\, t^{m+\alpha-\eta} \{t^\eta L(t)\}\, dt$$

where $0 < \eta < \frac{1}{2}$. Since $m = 1, 2, \ldots, \delta$, it is by (24) $m + \alpha - \eta > 0$, and therefore

$$|H_N^{(m)}(x)| \leqslant (1 + \varepsilon_1^{(m)}(N))\, x^{m+\alpha-\theta-\eta} \int_N^{t_1^{(m)}} w^{(m)}(t) \max_{0 \leqslant \xi \leqslant t} \{\xi^\eta L(\xi)\}\, dt$$

$$\leqslant (1 + \varepsilon^{(m)}(N))\, x^{m+\alpha-\theta-\eta} \max_{0 \leqslant \xi \leqslant x} \{\xi^\eta L(\xi)\} \int_N^{t_1^{(m)}} w^{(m)}(t)\, dt.$$

By the property (iv) of slowly oscillating functions we have

$$|H_N^{(m)}(x)| \leqslant \left(1+\varepsilon_1^{(m)}(N)\right) x^{m+\alpha-\theta-\eta}\, x^{\eta}\, L_1(x)\, w^{(m-1)}(t)\Big|_N^{t_1^{(m)}}$$

where

$$L_1(x) \cong L(x), \quad x \to \infty .$$

$L_1(x)$ is also a slowly oscillating function by the property (ii) of slowly oscillating functions. In virtue of (40) and the fact that the function $w^{(m-1)}(t)$ is increasing within the interval $(N, t_1^{(m)})$ we obtain

$$|H_N^{(m)}(x)| \leqslant C_1^{(m)}(\varkappa)\, \left(1+\varepsilon_1^{(m)}(N)\right) x^{\alpha+m(1-\theta)}\, L_1(x) \tag{44}$$

where

$$C_1^{(m)}(\varkappa) = [1-\exp(-j_1^{(m)})]^{\varkappa-m} \exp(-j_1^{(m)})\, P_{m-1}[\exp(-j_1^{(m)})].$$

Separately we have to evaluate the integral

$$H_N^{(0)}(x) = x^{-\theta} \int_N^x w(t, x)\, t^{\alpha} L(t)\, \lambda(t)\, dt.$$

Since $|\lambda(t)| \leqslant 1 + |\varepsilon(t)| \leqslant 1 + \varepsilon_*^{(0)}(N)$ for $N \leqslant t \leqslant x$, it follows

$$|H_N^{(0)}(x)| \leqslant \left(1+\varepsilon_*^{(0)}(N)\right) x^{-\theta} \int_N^x w(t, x)\, t^{\alpha-\eta} \max_{0\leqslant\xi\leqslant t} \{\xi^{\eta} L(\xi)\}\, dt$$

$$\leqslant \left(1+\varepsilon_*^{(0)}(N)\right) x^{-\theta} \max_{0\leqslant\xi\leqslant x} \{\xi^{\eta} L(\xi)\} \int_N^x w(t, x)\, t^{\alpha-\eta}\, dt.$$

By the property (iv) of slowly oscillating functions we have

$$|H_N^{(0)}(x)| \leqslant \left(1+\varepsilon_*^{(0)}(N)\right) x^{-\theta} \cdot x^{\eta} L_1(x) \int_N^x w(t, x)\, t^{\alpha-\eta}\, dt.$$

Since $\varkappa \geqslant \delta$, and δ is a positive integer, it follows $\varkappa \geqslant 1$, and $w(t, x) < 1$. Hence

$$|H_N^{(0)}(x)| \leqslant \left(1+\varepsilon_*^{(0)}(N)\right) x^{-\theta+\eta} L_1(x)\, M_0\, t^{\alpha-\eta+1}\Big|_N^x,$$

where $M_0 = 1/(\alpha-\eta+1)$. According to (24) $\alpha-\eta+1>0$ for $0<\eta<\frac{1}{2}$, therefore

$$|H_N^{(0)}(x)| \leqslant \left(1+\varepsilon_*^{(0)}(N)\right) M_0\, x^{\alpha+1-\theta} L_1(x). \tag{45}$$

The integrals of the form $H_r^{(m)}(x)$ appear for $m = r, r+1, \ldots, \delta$, $(r = 2, 3, \ldots, s)$. Suppose that $w^{(m)}(t) > 0$ in the interval $(t_{r-1}^{(m)}, t_r^{(m)})$. Further $|\lambda(t)| \leqslant 1 + |\varepsilon(t)| \leqslant 1 + \varepsilon_r^{(m)}(N)$ for $t_{r-1}^{(m)} \leqslant t \leqslant t_r^{(m)}$, therefore

$$|H_r^{(m)}(x)| \leqslant \left(1+\varepsilon_r^{(m)}(N)\right) x^{-\theta} \int_{t_{r-1}^{(m)}}^{t_r^{(m)}} w^{(m)}(t)\, t^{m+\alpha-\eta} \{t^{\eta} L(t)\}\, dt.$$

According to (24) $m+\alpha-\eta>0$, therefore

$$|H_r^{(m)}(x)|\leqslant(1+\varepsilon_r^{(m)}(N))\,x^{m+\alpha-\theta-\eta}\int_{t_r^{(m)}}^{t_r^{(m)}} w^{(m)}(t)\max_{0\leqslant\xi\leqslant t}\{\xi^\eta L(\xi)\}\,dt$$

$$\leqslant(1+\varepsilon_r^{(m)}(N))\,x^{m+\alpha-\theta-\eta}\max_{0\leqslant\xi\leqslant x}\{\xi^\eta L(\xi)\}\,w^{(m-1)}(t)\Big|_{t_{r-1}^{(m)}}^{t_r^{(m)}}.$$

By the property (iv) of slowly oscillating functions and in virtue of (40) we have

(46) $$|H_r^{(m)}(x)|\leqslant C_r^{(m)}(\varkappa)\,(1+\varepsilon_r^{(m)}(N))\,x^{\alpha+m(1-\theta)}\,L_1(x)$$

where

$$C_r^{(m)}(\varkappa)=[1-\exp(-j_r^{(m)})]^{\varkappa-m}\exp(-j_r^{(m)})\,P_{m-1}[\exp(-j_r^{(m)})]-$$

$$-[1-\exp(-j_{r-1}^{(m)})]^{\varkappa-m}\exp(-j_{r-1}^{(m)})\,P_{m-1}[\exp(-j_{r-1}^{(m)})],$$

and

$$L_1(x)\cong L(x),\quad x\to\infty.$$

Still we have to evaluate the integral $H_*^{(m)}(x)$ for $m=1,2,\ldots,\delta$. The function $w^{(m)}(t)$ has the same sign in the interval $(t_s^{(m)},x)$, say, positive. Then the function $w^{(m-1)}(t)$ is increasing in this interval, and since $w^{(m-1)}(x)=0$, it is negative. Further $|\lambda(t)|\leqslant1+|\varepsilon(t)|\leqslant1+\varepsilon_*^{(m)}(N)$ for $t_s^{(m)}\leqslant t\leqslant x$, therefore

$$|H_*^{(m)}(x)|\leqslant(1+\varepsilon_*^{(m)}(N))\,x^{-\theta}\int_{t_s^{(m)}}^{x} w^{(m)}(t)\,t^{m+\alpha-\eta}\max_{0\leqslant\xi\leqslant t}\{\xi^\eta L(\xi)\}\,dt.$$

Since by (24) $m+\alpha-\eta>0$ $(m=1,2,\ldots,\delta)$, it follows

$$|H_*^{(m)}(x)|\leqslant(1+\varepsilon_*^{(m)}(N))\,x^{\alpha+m-\theta-\eta}\max_{0\leqslant\xi\leqslant x}\{\xi^\eta L(\xi)\}\,w^{(m-1)}(t)\Big|_{t_s^{(m)}}^{x}$$

i. e.

(47) $$|H_*^{(m)}(x)|\leqslant C_*^{(m)}(\varkappa)\,(1+\varepsilon_*^{(m)}(N))\,x^{\alpha+m(1-\theta)}\,L_1(x)$$

where

$$C_*^{(m)}(\varkappa)=-[1-\exp(-j_s^{(m)})]^{\varkappa-m}\exp(-j_s^{(m)})\,P_{m-1}[\exp(-j_s^{(m)})],$$

and

$$L_1(x)\cong L(x),\quad x\to\infty.$$

$C_*^{(m)}(\varkappa)$ is positive, as $w^{(m-1)}(t)<0$ within the interval $t_s^{(m)}\leqslant t<x$.

We can choose N such that $\varepsilon_\mu^{(m)}(N)\leqslant\varepsilon^{(m)}(N)$, $(\mu=1,2,\ldots,s,*)$ and in virtue of (42), (43), (44), (45), (46), (47) we have for $m=0,1,2,\ldots,\delta$ and $\varkappa\geqslant\delta$

(48) $$|F_m(x)|\leqslant K_0^{(m)}x^{-(m+1)\theta}\exp[(N-x)x^{-\theta}]+$$
$$+(1+\varepsilon^{(m)}(N))\,C^{(m)}(\varkappa)\,x^{\alpha+m(1-\theta)}L_1(x)$$

where

$$C^{(m)}(\varkappa)=\sum_{\nu=1}^{s}C_\nu^{(m)}(\varkappa)+C_*^{(m)}(\varkappa).$$

Being $t_\mu^{(m)} \cong x$, $x \to \infty$ $(\mu = 1, 2, \ldots, s)$, we can choose N arbitrarly large, so that $\varepsilon^{(m)}(N) \to 0$. Since

$$L_1(x) \cong L(x), \qquad x \to \infty$$

we have from (48)

(49) $$F_m(x) = O\{x^{\alpha + m(1-\theta)} L(x)\}, \quad x \to \infty \quad (m = 0, 1, 2, \ldots \delta).$$

Finally, in virtue of (38) and (49) we obtain

$$G_\theta^\varkappa(S; x) = O\{x^{\alpha + \delta(1-\theta)} L(x)\}, \quad x \to \infty$$

for $\varkappa \geqslant \delta$ and the theorem 2 is proved.

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