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UNIFORM CONVERGENCE FACTORS OF ORTHOGONAL EXPANSIONS

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1. Let $\{\Phi_\nu(t)\}$, $\nu=0, 1, 2, \ldots,$ be an orthonormal system (ONS) in $[0, 1]$. Given an integrable function f, we write the expansion of f in the system $\{\Phi_\nu(t)\}$,

$$f \sim \Sigma\, c_\nu \Phi_\nu(t)$$

where

$$c_\nu = \int_0^1 f(t)\, \Phi_\nu(t)\, dt.$$

Let $\{\lambda_\nu\}$ be a sequence of real numbers. If the series

$$\sum_0^\infty \lambda_\nu c_\nu \Phi_\nu(t)$$

converges uniformly with respect to t for any function f of a given class, we say that $\{\lambda_\nu\}$ is a sequence of uniform convergence factors of the orthonormal expansions of functions of this class.

We define the operator $T_{(n,t)}$ by

$$T_{(n,t)} f = \sum_{\nu=0}^{n} \lambda_\nu c_\nu \Phi_\nu(t) = \int_0^1 f(u) \left(\sum_{\nu=0}^{n} \lambda_\nu \Phi_\nu(t) \Phi_\nu(u) \right) du.$$

Karamata [2] and others have considered the uniform convergence factors of Fourier series. Aljančić [1] has studied the uniform convergence factors of general orthonormal expansions of functions of class C, the continuous functions. We will give necessary and sufficient conditions (NASC) that $\{\lambda_\nu\}$ be a sequence of uniform convergence factors for the orthonormal expansions of functions of various classes.

Our principal tool is a lemma of Banach—Steinhaus type, which we develop in § 2. We consider the functions of class C in § 3, giving two alternate results with different conditions on the ONS. In § 4 we treat the classes L^p, $1 \leqslant p \leqslant \infty$.

2. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two Banach spaces over the same scalar field. Let $\alpha=(n, t)$, where $n=0, 1, 2, \ldots$, and $\{t\}$ is any set I. For each α, T_α will be a linear operator from $\mathfrak{X}$ to $\mathfrak{Y}$. We will say that the sequence $\{T_\alpha\}$ *converges uniformly in t on $x' \subset x$* if for any $\varepsilon>0$ and any $\mathfrak{X} \subset \mathfrak{X}'$, there is an integer N such that

$$\| (T_{(n,t)} - T_{(m,t)}) x \|_{\mathfrak{Y}} < \varepsilon$$

for any $t \varepsilon I$ if $n, m>N$. If the reference to $\mathfrak{X}'$ is omitted, it will be understood that we are referring to uniform convergence on the whole space $\mathfrak{X}$.

It is clear that an analogous definition of a uniform limit operator T_t, of $\{T_\alpha\}$ can be given and that the existence of this limit operator is equivalent to the uniform convergence of the sequence.

We will establish the following result on uniform convergence.

Lemma. *$\{T_\alpha\}$ converges uniformly in t if and only if*

(A) *there exists an M and an N such that*

$$\| T_{(n,t)} - T_{(m,t)} \| < M$$

for any $t \varepsilon I$ whenever $n, m > N$,

(B) *$\{T_\alpha\}$ converges uniformly in t on $\mathfrak{X}'$, a dense subset of $\mathfrak{X}$.*

Proof. Let us suppose that condition (A) is not satisfied. Then there exists $\{n_i\}$ and $\{m_i\}$ tending to $+\infty$ with i, and a sequence $\{t_i\}$, such that

$$\lim_{i \to \infty} \| T_{(n_i, t_i)} - T_{(m_i, t_i)} \| = +\infty .$$

If $\{T_\alpha\}$ converges uniformly in t, then $\lim\limits_{i\to\infty} (T_{(n_i, t_i)} - T_{(m_i, t_i)}) x = 0$. Applying the Banach—Steinhaus theorem, we find that $\| T_{(n_i, t_i)} - T_{(m_i, t_i)}) \|$ is bounded uniformly in i, which contradicts our assumption.

The sufficiency of (A) and (B) is clear.

3. In our study of the expansions of functions of class C, it should be noted that we do *not* assume that the Φ_ν's are continuous or even bounded. We present two results of rather different character.

Theorem 1. *Let $\{\Phi_\nu\}$ be an* ONS *such that the set of $f \varepsilon C$ for which $\sum\limits_0^\infty \lambda_\nu c_\nu \Phi_\nu$ is terminating is dense in C. The* NASC *that $\{\lambda_\nu\}$ be a sequence of uniform convergence factors of functions of class C is that there exists n_0, M such that*

$$(*) \qquad \int_0^1 \left| \sum_{n_0}^{n} \lambda_\nu \Phi_\nu(t) \Phi_\nu(u) \right| du < M$$

for all t and large n.

Proof. Condition (B) of our basic theorem is equivalent to the hipothesis on $\{\Phi_\nu\}$.

Condition (A) implies that there exist N, M such that for $n>m>N$,

$$\int_0^1 \left| \sum_{m+1}^{n} \lambda_\nu \Phi_\nu(t) \Phi_\nu(u) \right| du = \| T_{(n,t)} - T_{(m,t)} \| < M,$$

and for fixed m this is $(*)$.

If (*) is satisfied, then for n, $m > n_0$,

$$\| T_{(n,\, t)} - T_{(m,\, t)} \| \leqslant \int_0^1 \left| \sum_{n_0}^{n} \lambda_\nu \Phi_\nu(t) \Phi_\nu(u) \right| du + \int_0^1 \left| \sum_{n_0}^{m} \lambda_\nu \Phi_\nu(t) \Phi_\nu(u) \right| du < 2M$$

which is condition (A).

We observe that (*) implies that $\Phi_\nu(t)$ is a bounded function for large ν and $\lambda_\nu \neq 0$ since

$$| \Phi_\nu(t) | = \frac{1}{|\lambda_\nu| \int_0^1 |\Phi_\nu(u)|\, du} \cdot \| T_{(\nu,\, t)} - T_{(\nu-1,\, t)} \|$$

$$< 2M \Big/ \left(|\lambda_\nu| \int_0^1 |\Phi_\nu(u)|\, du \right).$$

It is interesting to note that although the Haar expansion of any $f \varepsilon C$ converges uniformly to f, the only functions in C with terminating expansions are constants. This suggests the following alternate theorem.

Theorem 2. *The conclusion of theorem* 1 *is valid if* $\{\Phi_\nu\}$ *is an* ONS *such that all* $f \varepsilon C$ *for which* $\Sigma \lambda_\nu c_\nu \Phi_\nu$ *is non-terminating may be uniformly approximated by finite linear combinations of functions belonging to an* ONS *containing* $\{\Phi_\nu\}$.

Proof. Let us suppose that we have enlarged $\{\Phi_\nu\}$ by adjoining the remainder of the ONS described above, and that we have inserted zeros into our factor sequence at the places corresponding to the adjoined functions.

Let Φ be the space of bounded finite linear combinations of Φ_ν's. Let P be the completion under the sup norm of the direct sum of C and Φ. Since $\{T_{(n,\, t)}\}$ converges uniformly on Φ and the $f \varepsilon C$ for which $\Sigma \lambda_\nu c_\nu \Phi_\nu$ is terminating, we see that condition (B) is satisfied. Since $\| T_{(n,\, t)} \|_P = \| T_{(n,\, y)} \|_C$, the argument may be concluded as in the proof of theorem 1.

4. We turn now to the functions of class L^p, $p \geqslant 1$. We differentiate between the cases $1 \leqslant p \leqslant 2$, and $p > 2$, since if $\{\Phi_\nu\}$ is an ONS in L^q, $\frac{1}{p} + \frac{1}{q} = 1$, then for $1 \leqslant p \leqslant 2$, $\{\Phi_\nu\}$ can be extended to a system closed in L^p.

It is to be noted again that our results imply that $\Phi_\nu(t)$ is a bounded function for large ν and $\lambda_\nu \neq 0$.

We state the following theorems without proof.

Theorem 3. *The* NASC *that* $\{\lambda_\nu\}$ *be a sequence of uniform convergence factors of the orthonormal expansions of functions of class* L^p, $1 \leqslant p \leqslant 2$, *is that there exist* n_0, M *such that*

$$\int_0^1 \left| \sum_{n_0}^{n} \lambda_\nu \Phi_\nu(t) \Phi_\nu(u) \right|^q du < M$$

for all t *and sufficiently large* n.

Theorem 4. *The conclusion of theorem* 3 *is valid for the class* L^p, $2 < p < \infty$, *if* $\{\Phi_\nu\}$ *can be extended to an* ONS *closed in* L^p.

The case of L^∞ cannot be approached in the same manner, since no countable system of functions is closed there. The lemma may be applied to yield the following result.

Theorem 5. *The* NASC *that* $\{\lambda_\nu\}$ *be a sequence of uniform convergence factors of the expansions of functions of class* L^∞ *in the* ONS $\{\Phi_\nu\}$ *are*

(a) *that there exist* n_0, M *such that*

$$\int_0^1 \left| \sum_{n_0}^{n} \lambda_\nu \Phi_\nu(t) \Phi_\nu(u) \right| du < M$$

for all t and large n,

(b) $$\int_H \left(\sum_{m}^{n} \lambda_\nu \Phi_\nu(t) \Phi_\nu(u) \right) du \to 0$$

as $n, m \to \infty$, *uniformly in t for any measurable set* $H \subset [0, 1]$.

REFERENCES

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