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DUAL INTEGRAL EQUATIONS

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In this paper we shall examine the conditions under which the integral equations

$$
\begin{aligned}
&\int_0^{+\infty} x^\alpha f(x)\, J_\nu(yx)\, dx = U(y) \qquad (0 < y < 1),\\
&\int_0^{+\infty} f(x)\, J_\nu(yx)\, dx = V(y) \qquad (y > 1),
\end{aligned}
\tag{1}
$$

can be reduced to:

$$
\int_0^{+\infty} x^{\frac{\alpha}{2}} f(x)\, J_{\nu+\frac{\alpha}{2}}(yx)\, dx = \begin{cases} g_1(y) & (0<y<1)\\ g_2(y) & (y>1). \end{cases}
\tag{2}
$$

Then the function $f(x)$ can be computed by use of the inversion of the Hankel transform.

We employ the following notations:

$$
\underbrace{y^{-1}\frac{d}{dy}}_{k}\ \underbrace{y^{-1}\frac{d}{dy}}_{k-1}\cdots\ \underbrace{y^{-1}\frac{d}{dy}}_{1}\varphi(y) = \left(y^{-1}\frac{d}{dy}\right)^k \varphi(y)
$$

$$
I_f(y,\ \eta_1,\ \eta_2,\ \mu,\ k) = \int_{\eta_1}^{\eta_2} x^\mu f(x)\, J_k(yx)\, dx,
$$

$\varepsilon(\eta_1,\ \eta_2)$ denotes a function which is bounded for $0 \leqslant \eta_1 < \eta_2 \leqslant \infty$ and leads to 0 when $\eta_1 \to \eta_2$.

Theorem 1. *Assumptions:*

1. *We choose the natural numbers m and* $0<\varepsilon_1<1$ *in the way that the inequalities:* $-2(m-\varepsilon_1)<\alpha<0,\ \nu+\frac{\alpha}{2}+1>0,\ \nu+\alpha+\frac{3}{2}>0, \nu>-1$ *are satisfied.*

2. *The solution of* (1) *exists and is a finite function for* $0<x<\infty$, *satisfying the conditions:*

For $0<y<1$

$$|I_f(y, \eta_1, \eta_2, \alpha-a, \nu+a)| \begin{cases} <\infty \ for \ a=\frac{\alpha}{2}+m-\varepsilon_1, \quad \eta_1=0, \quad \eta_2=\infty \\ \leqslant y^\beta \varepsilon(\eta_1, \eta_2) \ for \ a=0, \ a=\frac{\alpha}{2}, \ \beta>-\nu-a-2 \end{cases}$$

For $y>1$

$$|I_f(y, \eta_1, \eta_2, 0, \nu)| \leqslant y^\gamma \varepsilon(\eta_1, \eta_2) \ for \ \gamma<\nu+\alpha.$$

Statement: system (1) *can be reduced to*

$$(3) \qquad \int_0^{+\infty} x^{\frac{\alpha}{2}} f(x)\, J_{\nu+\frac{\alpha}{2}}(yx)\, dx = \begin{cases} \dfrac{2^{2-m-\frac{\alpha}{2}} y^{-\nu-\frac{\alpha}{2}-1}}{\Gamma(\varepsilon_1)\Gamma\left(m-\varepsilon_1+\frac{\alpha}{2}\right)} \dfrac{d}{dy} \displaystyle\int_0^y \frac{t\,\psi_1(t)\,dt}{(y^2-t^2)^{1-\varepsilon_1}} & (0<y<1) \\ \dfrac{2^{1+\frac{\alpha}{2}} y^{-\frac{\alpha}{2}}}{\Gamma\left(-\frac{\alpha}{2}\right)} \displaystyle\int_1^{+\infty} \frac{(\tau^2-1)^{-\frac{\alpha}{2}-1}}{\tau^{\nu-1}} V(y\tau)\,d\tau & (y>1) \end{cases}$$

where:

$$\psi_1(y) = \left(y^{-1}\frac{d}{dy}\right)^{m-1} \int_0^y t^{\nu+1}(y^2-t^2)^{\frac{\alpha}{2}+m-\varepsilon_1-1}\, U(t)\,dt.$$

Theorem II. *Assumptions:*

1. *We choose the natural numbers* n *and* $0<\varepsilon_2<1$ *in the way that the inequalites:* $0<\alpha<2(n-\varepsilon_2)$, $\nu+\frac{\alpha}{2}-2(n-\varepsilon_2)+\frac{3}{2}>0$, $\nu>-1$.

2. *The solution of* (1) *exists and is a finite function for* $0<x<\infty$ *satisfying the conditions*

For $0<y<1$

$$|I_f(y, \eta_1, \eta_2, \alpha, \nu)| \leqslant y^\delta \varepsilon(\eta_1, \eta_2) \ for \ \delta>-\nu-2.$$

For $y>1$

$$|I_f(y, \eta_1, \eta_2, b, \nu+b)| \begin{cases} <\infty \ for \ b=\frac{\alpha}{2}-n+\varepsilon_2, \ \eta_1=0, \ \eta_2=\infty \\ \leqslant y^\sigma \varepsilon(\eta_1, \eta_2) \ for \ b=0, \ b=\frac{\alpha}{2}, \ \sigma<\nu+\alpha-2(n-\varepsilon_2)-b. \end{cases}$$

Statement: system (1) *can be reduced to*

$$(4) \qquad \int_0^{+\infty} x^{\frac{\alpha}{2}} f(x)\, J_{\nu+\frac{\alpha}{2}}(yx)\, dx =$$

$$= \begin{cases} \dfrac{y^{\frac{\alpha}{2}}}{2^{\frac{\alpha}{2}-1}\Gamma\left(\frac{\alpha}{2}\right)} \displaystyle\int_0^1 \tau^{\nu+1}(1-\tau^2)^{\frac{\alpha}{2}-1}\, U(y\tau)\,d\tau & (0<y<1), \\ -\dfrac{2^{2-n+\frac{\alpha}{2}} y^{\nu+\frac{\alpha}{2}-1+2\varepsilon_2}}{\Gamma(\varepsilon_2)\Gamma\left(n-\varepsilon_2-\frac{\alpha}{2}\right)} \dfrac{d}{dy} y^{2-2\varepsilon_2} \displaystyle\int_y^{+\infty} \frac{t^{-1}\psi_2(t)\,dt}{(t^2-y^2)^{1-\varepsilon_2}} & (y>1), \end{cases}$$

where

$$\psi_2(y)=\left(y^{-1}\frac{d}{dy}\right)^{n-1}\int\limits_y^{+\infty}\frac{(t^2-y^2)^{-\frac{\alpha}{2}+n-\varepsilon_2-1}}{t^{\nu-1}}V(t)\,dt.$$

Remark: If in the theorem I we put $\varepsilon_1=0$ and instead of the right side of the first equation of system (3) we take the expression

$$\frac{2^{1-m-\frac{\alpha}{2}}y^{-\nu-\frac{\alpha}{2}}}{\Gamma\left(m+\frac{\alpha}{2}\right)}\left(y^{-1}\frac{d}{dy}\right)^m\int\limits_y^{+\infty}t^{\nu+1}(y^2-t^2)^{\frac{\alpha}{2}+m-\varepsilon_1-1}U(t)\,dt$$

theorem I is still valid.

If in the theorem II we put $\varepsilon_2=0$ and instead of the right side of the second equation of system (4) we take the expression:

$$\frac{2^{1-n+\frac{\alpha}{2}}y^{\nu+\frac{\alpha}{2}}}{\Gamma\left(n-\frac{\alpha}{2}\right)}\left(y^{-1}\frac{d}{dy}\right)^n\int\limits_y^{+\infty}\frac{(t^2-y^2)^{-\frac{\alpha}{2}+n-\varepsilon_2-1}}{t^{\nu-1}}V(t)\,dt$$

theorem II is still valid.

In this way we can simplify formal expressions, but we restrict the domains of the indices.

We prove first some lemmata:

Lemma I. *Suppose that*

1. $F(x,\tau,y)=\varphi(x)\,\tau^{k_1-\mu_1+1}(y^2-\tau^2)^{\mu_1-1}J_{k_1-\mu_1}(\tau x)\qquad(0<y<1)$, $k_1+1>\mu_1>0$ *and* $\varphi(x)$ *finite for* $0<x<\infty$.

2.

$$|I_\varphi(y,\eta_1,\eta_2,c-\mu_1,k_1-c)|\begin{cases}<\infty & \text{for } c=0,\ \eta_1=0,\ \eta_2=\infty\\ \leqslant y^h\,\varepsilon(\eta_1,\eta_2) & \text{for } c=\mu_1,\ h>\mu_1-k_1-2\end{cases}$$

then

$$(5)\qquad \int\limits_0^y d\tau\int\limits_0^{+\infty}F(x,\tau,y)\,dx=\int\limits_0^{+\infty}dx\int\limits_0^y F\,d\tau=\frac{2^{\mu_1-1}\Gamma(\mu_1)}{y^{-k_1}}\int\limits_0^{+\infty}x^{-\mu_1}\varphi(x)J_{k_1}(yx)\,dx$$

Lemma II. *Suppose that*

1. $G(x,\tau,y)=g(x)\,\tau^{1-k_2-\mu_2}(\tau^2-y^2)^{\mu_2-1}J_{k_2+\mu_2}(x\tau)$, $y>1$, $k_2+\frac{3}{2}>\mu_2>0$ *and* $g(x)$ *finite for* $0<x<\infty$.

2.

$$|I_g(y,\eta_1,\eta_2,e-\mu_2,k_2+e)|\begin{cases}<\infty & \text{for } e=0,\ \eta_1=0,\ \eta_2=\infty\\ \leqslant y^s\,\varepsilon(\eta_1,\eta_2) & \text{for } e=\mu_2,\ s<k_2-\mu_2\end{cases}$$

then

$$(6)\qquad \int\limits_y^{+\infty}d\tau\int\limits_0^{+\infty}G(x,\tau,y)\,dx=\int\limits_0^{+\infty}dx\int\limits_y^{+\infty}G\,d\tau=\frac{2^{\mu_2-1}\Gamma(\mu_2)}{y^{k_2}}\int\limits_0^{+\infty}x^{-\mu_2}g(x)J_{k_2}(yx)\,dx.$$

Since lemma I and lemma II can be proved by the same method, we shall prove only lemma II.

First we split the integral:

$$\int\limits_0^{+\infty} dx \int\limits_y^{+\infty} G(x,\tau,y)\,d\tau = \int\limits_0^{1} dx \int\limits_y^{2y} G\,d\tau + \int\limits_0^{1} dx \int\limits_{2y}^{+\infty} G\,d\tau +$$

$$+\int\limits_1^{+\infty} dx \int\limits_y^{2y} G\,d\tau + \int\limits_1^{+\infty} dx \int\limits_{2y}^{+\infty} G\,d\tau = I_1 + I_2 + I_3 + I_4 .$$

Using a known theorem T. J. I'in Bromwich* we shall show that in every I_k, $k=1, 2, 3, 4$ it is possible to change the order of integration.

First, as for $\eta \geqslant 1$ and $0 < x < \infty$ the Bessel function $J_p(x)$ satisfies the evaluation:

$$J_p(x) \leqslant M \frac{x^p}{1+x^{p+\frac{1}{2}}} \qquad (M = \text{const.})$$

we hawe:

$$\left| g(x) \int\limits_y^{y+\frac{y}{\eta}} \tau^{1-k_2-\mu_2} (\tau^2-y^2)^{\mu_2-1} J_{k_2+\mu_2}(x\tau)\,d\tau \right| < \frac{M\cdot 3^{\mu_2+1}\, y^{2\mu_2} x^{k_2+\mu_2} |g(x)|}{\mu_2\, \eta^{\mu_2}}$$

$$\left| g(x) \int\limits_{2y\eta}^{+\infty} \tau^{1-k_2-\mu_2} (\tau^2-y^2)^{\mu_2-1} J_{k_2+\mu_2}(x\tau)\,d\tau \right|$$

by partial integration we obtain

$$\left| g(x) \int\limits_{2y\eta}^{+\infty} \tau^{1-k_2-\mu_2} (\tau^2-y^2)^{\mu_2-1} J_{k_2+\mu_2}(x\tau)\,d\tau \right|$$

$$= \left| x^{-1} g(x) \left\{ \tau^{1-k_2-\mu_2} (\tau^2-y^2)^{\mu_2-1} J_{k_2+\mu_2+1}(x\tau) \right|_{2y\eta}^{\infty} \right.$$

$$\left. -2 \int\limits_{2y\eta}^{+\infty} \tau^{-k_2-\mu_2} (\tau^2-y^2)^{\mu_2-2} [(\mu_2-1)\tau^2-(k_2+\mu_2)(\tau^2-y^2)] J_{k_2+\mu_2+1}(x\tau)\,d\tau \right|$$

$$< \frac{2M(26\mu_2+22|k_2|+25)\, x^{-\frac{2}{3}} |g(x)|}{9\left(k_2-\mu_2+\frac{3}{2}\right)(2y\eta)^{k_2-\mu_2+\frac{3}{2}}} .$$

For $\omega > 1$ and $0 \leqslant \eta_1 < \eta_2 \leqslant \infty$, using the condition 2. we get:

$$\left| \int\limits_{y+\frac{y}{\omega}}^{2y} \tau^{1-k_2-\mu_2} (\tau^2-y^2)^{\mu_2-1} d\tau \int\limits_{\eta_1}^{\eta_2} g(x) J_{k_2+\mu_2}(x\tau)\,dx \right| <$$

$$< 3^{\mu_2+1} y^{s-k_2} \varepsilon(\eta_1,\eta_2) \int\limits_y^{2y} (\tau-y)^{\mu_2-1} d\tau = \frac{3^{\mu_2+1}\varepsilon(\eta_1,\eta_2)}{\mu_2\, y^{k_2-\mu_2-s}}$$

$$\left| \int\limits_{2y}^{\omega} \tau^{1-k_2-\mu_2} (\tau^2-y^2)^{\mu_2-1} d\tau \int\limits_{\eta_1}^{\eta_2} g(x) J_{k_2+\mu_2}(x\tau)\,dx \right| < \varepsilon(\eta_1,\eta_2) \int\limits_{2y}^{+\infty} \tau^{\mu_2-k_2+s-1} d\tau$$

$$= \frac{\varepsilon(\eta_1,\eta_2)}{(k_2-\mu_2-s)(2y)^{k_2-\mu_2-s}} .$$

* T. J. I'A. Bromwich: *Note on double limits and on the inversion of a repeated infinite integrale.* Proc. London Math. Soc. (2) 1 (1904), 176—201.

As for fixed $\xi > 1$ the integral

$$\int\limits_{\frac{1}{\xi}}^{1} x^{k_2+\mu_2} |g(x)|\, dx, \quad \int\limits_{1}^{\xi} x^{k_2+\mu_2} |g(x)|\, dx, \quad \int\limits_{\frac{1}{\xi}}^{1} x^{-\frac{3}{2}} |g(x)|\, dx, \quad \int\limits_{1}^{\xi} x^{-\frac{3}{2}} |g(x)|\, dx$$

exist and $\varepsilon(\eta_1, \eta_2) \to 0$ for $\eta_1 \to \eta_2$, we conclude that one can change the order of integration in every I_k $(k = 1, 2, 3, 4)$.

For $k_2 + \frac{3}{2} > \mu_2 > 0$ and $y > 1$ we have

$$x^{-\mu_2} J_{k_2}(yx) = \frac{y^{k_2}}{2^{\mu_2-1}\Gamma(\mu_2)} \int\limits_{y}^{+\infty} \tau^{1-k_2-\mu_2} (\tau^2 - y^2)^{\mu_2-1} J_{k_2+\mu_2}(\tau x)\, d\tau.*$$

Lemmata II is proved.

L e m m a III. *Suppose that*

$$\left| \int\limits_0^{+\infty} x^{\frac{\alpha}{2}} f(x) J_{\nu+\frac{\alpha}{2}}(\tau x)\, dx \right| \leqslant \tau^a R$$

for $0 < \tau < 1$ *and* $a > -\nu - \frac{\alpha}{2} - 2$, $R = const.$

Then the integral

$$\int\limits_0^y \tau^{\nu+\frac{\alpha}{2}+1} (y^2 - \tau^2)^{r-\frac{\alpha}{2}-\lambda}\, d\tau \int\limits_0^{+\infty} x^{\frac{\alpha}{2}} f(x) J_{\nu+\frac{\alpha}{2}}(\tau x)\, dx \tag{7}$$

is uniformly convergent for $0 < \delta \leqslant y \leqslant 1$ *and* $\lambda = 1, 2, \ldots, (m-1)$, $(m-1) < r - \frac{\alpha}{2} + 1 \leqslant m$.

L e m m a IV. *Suppose that*

$$\left| \int\limits_0^{+\infty} x^{\frac{\alpha}{2}} f(x) J_{\nu+\frac{\alpha}{2}}(\tau x)\, dx \right| \leqslant \tau^b R, \quad \tau > 1, \ b < \nu - \frac{\alpha}{2} - 2p - 2, \quad R = const.$$

Then the integral

$$\int\limits_y^{+\infty} \tau^{-\nu-\frac{\alpha}{2}+1} (\tau^2 - y^2)^{p+\frac{\alpha}{2}-\lambda}\, d\tau \int\limits_0^{+\infty} x^{\frac{\alpha}{2}} f(x) J_{\nu+\frac{\alpha}{2}}(\tau x)\, dx \tag{8}$$

is uniformly convergent for $1 \leqslant y \leqslant \Delta < \infty$ *and* $\lambda = 1, 2, \ldots, (n-1)$, $(n-1) < p + \frac{\alpha}{2} + 1 \leqslant n$.

Since lemma III and lemma IV can be proved by the same method, we shall prove only lemma IV.

In the integral (8) put $\tau = yt$, $d\tau = y\,dt$ so:

$$y^{-\nu+\frac{\alpha}{2}+2p-2\lambda+2} \int\limits_1^{+\infty} t^{-\nu-\frac{\alpha}{2}+1} (t^2-1)^{p+\frac{\alpha}{2}-\lambda}\, dt \int\limits_0^{+\infty} x^{\frac{\alpha}{2}} f(x) J_{\nu+\frac{\alpha}{2}}(ytx)\, dx$$

$$\left| y^{-\nu+\frac{\alpha}{2}+2p-2\lambda+2} t^{-\nu-\frac{\alpha}{2}+1} (t^2-1)^{p+\frac{\alpha}{2}-\lambda} \int\limits_0^{+\infty} x^{\frac{\alpha}{2}} f(x) J_{\nu+\frac{\alpha}{2}}(ytx)\, dx \right|$$

$$\leqslant R y^{-\nu+\frac{\alpha}{2}+2p-2\lambda+b+2} t^{-\nu-\frac{\alpha}{2}+b+1} (t^2-1)^{p+\frac{\alpha}{2}-\lambda} < R t^{-\nu-\frac{\alpha}{2}+b+1} (t^2-1)^{p+\frac{\alpha}{2}-\lambda}.$$

* G. N. W a t s o n: *A treatise on the theory of Bessel functions.* Cambridge 1948.

Since the integral: $\int_1^{+\infty} t^{-\nu-\frac{\alpha}{2}+b+1}(t^2-1)^{p+\frac{\alpha}{2}-\lambda}\,dt$ is convergent, according the criterion of Weierstrass it follows that (8) converges uniformly.

Proof of the theorem I *and the theorem* II.

We intrcduce a parametar τ in the system (1) as to be able to write it in the form:

$$(9)\qquad \begin{aligned} &\int_0^{+\infty} x^{\alpha} f(x)\, J_\nu(\tau y x)\, dx = U(\tau y) \qquad (0<y<1,\ 0<\tau<1). \\ &\int_0^{+\infty} f(x)\, J_\nu(\tau y x)\, dx = V(\tau y) \qquad (y>1,\ \tau>1). \end{aligned}$$

By suitable multiplication and integration we obtain:

$$(10)\qquad \begin{aligned} &\frac{y^{r+1}}{2^r\,\Gamma(r+1)}\int_0^1 \tau^{\nu+1}(1-\tau^2)^r\,d\tau \int_0^{+\infty} x^{\alpha} f(x)\, J_\nu(y\tau x)\,dx \\ &\qquad = \frac{y^{r+1}}{2^r\,\Gamma(r+1)}\int_0^1 \frac{(1-\tau^2)^r}{\tau^{-\nu-1}}\,U(y\tau)\,d\tau, \\ &\frac{y^{p+1}}{2^p\,\Gamma(p+1)}\int_1^{+\infty} \frac{(\tau^2-1)^p}{\tau^{\nu-1}}\,d\tau \int_0^{+\infty} f(x)\, J_\nu(y\tau x)\,dx = \frac{y^{p+1}}{2^p\,\Gamma(p+1)}\int_1^{+\infty} \frac{(\tau^2-1)^p}{\tau^{\nu-1}}\,V(y\tau)\,d\tau, \end{aligned}$$

where r and p are arbitrary parameters.

Put now: $\varphi(x)=x^{\alpha} f(x)$, $\mu_1=r+1$, $k_1=\nu+r+1$, $g(x)=f(x)$, $\mu_2=p+1$, $k_2=\nu-p-1$ and suppose at first that these functions and parameters do satisfy all the conditions of lemma I and lemma II.

Then by the lemmata the system (10) can by reduced to:

$$(11)\qquad \begin{aligned} &\int_0^{+\infty} x^{\alpha-r-1} f(x)\, J_{\nu+r+1}(yx)\,dx = \frac{y^{-\left(r+\nu+\frac{1}{2}\right)}}{2^r\,\Gamma(r+1)}\,\theta(y) \qquad (0<y<1), \\ &\int_0^{+\infty} x^{-p-1} f(x)\, J_{\nu-p-1}(yx)\,dx = \frac{y^{\nu-p-\frac{1}{2}}}{2^p\,\Gamma(p+1)}\,\omega(y) \qquad (y>1), \end{aligned}$$

where

$$\theta(y) = \int_0^y \frac{(y^2-t^2)^r}{t^{-\nu-1}}\,U(t)\,dt \quad \text{and} \quad \omega(y) = \int_y^{+\infty} \frac{(t^2-y^2)^p}{t^{\nu-1}}\,V(t)\,dt.$$

Putting: $\varphi(x) = x^{\frac{\alpha}{2}} f(x)$, $\mu_1 = r-\frac{\alpha}{2}+1$, $k_1=\nu+r+1$, $g(x)=x^{\frac{\alpha}{2}} f(x)$, $\mu_2=p+\frac{\alpha}{2}+1$, $k_2=\nu-p-1$, and supposing that these functions and parameters do satisfy all the conditions of lemma I and lemma II system (11) by the lemmata can be reduced to:

$$(12)\quad \begin{aligned} &\int_0^y \tau^{\nu+\frac{\alpha}{2}+1} (y^2-\tau^2)^{r-\frac{\alpha}{2}} h\left(\tau, \nu+\frac{\alpha}{2}\right) d\tau = \frac{\Gamma\left(r+1-\frac{\alpha}{2}\right)}{2^{\frac{\alpha}{2}} \Gamma(r+1)} \theta(y) \\ &\int_y^{+\infty} \tau^{-\nu-\frac{\alpha}{2}+1} (\tau^2-y^2)^{p+\frac{\alpha}{2}} h\left(\tau, \nu+\frac{\alpha}{2}\right) d\tau = \frac{\Gamma\left(p+1+\frac{\alpha}{2}\right)}{2^{-\frac{\alpha}{2}} \Gamma(p+1)} \omega(y) \end{aligned}$$

where

$$h\left(\tau, \nu+\frac{\alpha}{2}\right) = \int_0^{+\infty} x^{\frac{\alpha}{2}} f(x) J_{\nu+\frac{\alpha}{2}}(\tau x)\, dx.$$

For r and p we introduce the conditions:

$$(13)\quad (n-1) < p+\frac{\alpha}{2}+1 \leqslant n, \qquad (m-1) < r-\frac{\alpha}{2}+1 \leqslant m.$$

Then by lemma III and lemma IV we can differentiale the first equation of syst m (12) $(m-1)$ times and the second $(n-1)$ times. So we obtain:

$$(14)\quad \begin{aligned} &\int_0^{+\infty} \tau^{\nu+\frac{\alpha}{2}+1} (y^2-\tau^2)^{r-\frac{\alpha}{2}+1-m} h\left(\tau, \nu+\frac{\alpha}{2}\right) d\tau \\ &\qquad = \frac{\Gamma\left(r+1-\frac{\alpha}{2}\right)}{2^{m-1+\frac{\alpha}{2}} \Gamma(r+1) \prod_{k=0}^{m-1}\left(r-\frac{\alpha}{2}+1-k\right)} \left(y^{-1}\frac{d}{dy}\right)^{m-1} \theta(y) \\ &\int_y^{+\infty} \tau^{-\nu-\frac{\alpha}{2}+1} (\tau^2-y^2)^{p+\frac{\alpha}{2}+1-n} h\left(\tau, \nu+\frac{\alpha}{2}\right) d\tau \\ &\qquad = \frac{\Gamma\left(p+1+\frac{\alpha}{2}\right)}{2^{n-1+\frac{\alpha}{2}} \Gamma(p+1) \prod_{k=0}^{n-1}\left(p+\frac{\alpha}{2}+1-k\right)} \left(y^{-1}\frac{d}{dy}\right)^{n-1} \omega(y). \end{aligned}$$

By means of inequalities (13) we can put $r+1-\frac{\alpha}{2}=m-\varepsilon_1$, $p+1+\frac{\alpha}{2}=n-\varepsilon_2$, $0 \leqslant \varepsilon_1 < 1$, $0 \leqslant \varepsilon_2 < 1$. Substituting parameters $r=m+\frac{\alpha}{2}-1-\varepsilon_1$, $p=n-\frac{\alpha}{2}-1-\varepsilon_2$ in the system (14) we obtain:

$$(15)\quad \begin{aligned} &\int_0^{y} \tau^{\nu+\frac{\alpha}{2}+1} (y^2-\tau^2)^{-\varepsilon_1} h\left(\tau, \nu+\frac{\alpha}{2}\right) d\tau \\ &\qquad = \frac{\Gamma(1-\varepsilon_1)}{2^{m-1+\frac{\alpha}{2}} \Gamma\left(m-\varepsilon_1+\frac{\alpha}{2}\right)} \left(y^{-1}\frac{d}{dy}\right)^{m-1} \theta(y) = R_1 \psi_1(y), \\ &\int_y^{+\infty} \tau^{-\nu-\frac{\alpha}{2}} (\tau^2-y^2)^{-\varepsilon_2} h\left(\tau, \nu+\frac{\alpha}{2}\right) d\tau \\ &\qquad = \frac{\Gamma(1-\varepsilon_2)}{2^{n-1-\frac{\alpha}{2}} \Gamma\left(n-\varepsilon_2-\frac{\alpha}{2}\right)} \left(y^{-1}\frac{d}{dy}\right)^{n-1} \omega(y) = R_2 \psi_2(y). \end{aligned}$$

For $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$ we hawe therefore:

$$(16)\qquad h\left(y, \nu+\frac{\alpha}{2}\right)=\begin{cases}\dfrac{y^{-\nu-\frac{\alpha}{2}}}{2^{m-1+\frac{\alpha}{2}}\,\Gamma\left(m+\frac{\alpha}{2}\right)}\left(y^{-1}\dfrac{d}{dy}\right)^m \theta(y) & (0<y<1),\\[2ex] \dfrac{y^{\nu+\frac{\alpha}{2}}}{2^{n-1-\frac{\alpha}{2}}\,\Gamma\left(n-\frac{\alpha}{2}\right)}\left(y^{-1}\dfrac{d}{dy}\right)^n \omega(y) & (y>1).\end{cases}$$

For $0<\varepsilon_1<1$ and $0<\varepsilon_2<1$ each equation of the system (15) can be reduced to the Abel's integral equation. In order to show this fact we make the following substitutions in the first equation of the system (15): $\tau^2 = t$, $d\tau = \frac{1}{2}\, t^{-\frac{1}{2}}\, dt$, $y^2 = z$ and in the second: $\tau^2 = \frac{1}{t}$, $d\tau = -\frac{1}{2}\, t^{-\frac{3}{2}}\, dt$, $y^2 = \frac{1}{u}$

$$\int_0^z t^{\frac{\nu}{2}+\frac{\alpha}{4}}(z-t)^{-\varepsilon_1}\, h\left(\sqrt{t}, \nu+\frac{\alpha}{2}\right) dt = 2\, R_1\, \psi_1\left(\sqrt{z}\right) \qquad (0<z<1),$$

$$\int_0^u t^{\frac{\nu}{2}+\frac{\alpha}{4}-2+\varepsilon_1}(u-t)^{-\varepsilon_2}\, h\left(u^{-\frac{1}{2}}\ \nu+\frac{\alpha}{2}\right) dt = 2\, R_2\, u^{-\varepsilon_2}\, \psi_2\left(u^{-\frac{1}{2}}\right) \qquad (0<u<1)$$

$$(17)\qquad \begin{aligned} h\left(\sqrt{z}, \nu+\frac{\alpha}{2}\right) &= \frac{2\, R_1\, z^{-\frac{\nu}{2}-\frac{\alpha}{4}}}{\Gamma(\varepsilon_1)\,\Gamma(1-\varepsilon_2)}\, \frac{d}{dz}\int_0^z \frac{\psi_1\left(\sqrt{t}\right)}{(z-t)^{1-\varepsilon_1}}\, dt,\\ h\left(u^{-\frac{1}{2}}, \nu+\frac{\alpha}{2}\right) &= \frac{2\, R_2\, u^{-\frac{\nu}{2}-\frac{\alpha}{4}+2-\varepsilon_2}}{\Gamma(\varepsilon_2)\,\Gamma(1-\varepsilon_1)}\, \frac{d}{du}\int_0^u \frac{\psi_2\left(t^{-\frac{1}{2}}\right) t^{-\varepsilon_2}\, dt}{(u-t)^{1-\varepsilon_2}}. \end{aligned}$$

By the subtitutions: $t = \tau^2$, $dt = 2\,\tau\, d\tau$, $z = y^2$ in the first equations of the system (17) and the substitutions: $t = \tau^{-2}$, $dt = -2\,\tau^{-3}\, d\tau$, $u = y^{-2}$ in the second we obtain:

$$(18)\qquad h\left(y, \nu+\frac{\alpha}{2}\right)=\begin{cases}\dfrac{2^{2-m-\frac{\alpha}{2}}\, y^{-\nu-\frac{\alpha}{2}-1}}{\Gamma(\varepsilon_1)\,\Gamma\left(m-\varepsilon_1+\frac{\alpha}{2}\right)}\,\dfrac{d}{dy}\displaystyle\int_0^y \frac{\tau\,\psi_1(\tau)\, d\tau}{(y^2-\tau^2)^{1-\varepsilon_1}} & (0<y<1),\\[2ex] -\dfrac{2^{2-n+\frac{\alpha}{2}}\, y^{\nu+\frac{\alpha}{2}-1+2\varepsilon_2}}{\Gamma(\varepsilon_2)\,\Gamma\left(n-\varepsilon_2-\frac{\alpha}{2}\right)}\,\dfrac{d}{dy}\, y^{2-2\varepsilon_2}\displaystyle\int_y^{+\infty} \frac{\tau^{-1}\,\psi_2(\tau)\, d\tau}{(\tau^2-y^2)^{1-\varepsilon_2}} & (y>1).\end{cases}$$

The assumptions made in the proofs of our theorems can be justified by the assumptions in the theorem I and theorem II; therefore system (18) contains the proof of the first equation in the system (3) and the proof of the second equation in (4).

By substitutions $r = \frac{\alpha}{2} - 1$, $p = -\frac{\alpha}{2} - 1$ in the system (13) we obtain the other two equations of system (3) and (4). This proves the theorems.