PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 2 (16), 1962, p. 71—74.

COUPLE STRESS IN NON EUCLIDEAN CONTINUA

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(Received 19. XII 1962)

Incompatible deformations (caused e. g. by dislocations) produce, besides a stress field, a field of couple stress, the existing stress tensor in such cases is not symmetric (cf. [1], [2]). The deformed state of the material considered, i. e. the state to which stress and couple stress fields are refered, does not present a euclidean continuum, but can be considered as a continuum in a linearly connected space L_3, in general with torsion.

In a previuos paper [3] the equilibrium conditions were derived for internal stress in a material continuum, considered in a metric linearly connected space. The aime of the present note is the derivation of the conditions for couple stress in the same space.

In euclidean spaces moments are defined by means of position vectors, the notion which can not be directly generalized since the „position vector" is a privilege of euclidean spaces. Instead of the usual definition of moments we shall adopt here another one, suggested in [4] for euclidean spaces and independent of the notion „position vector". The derived equilibrium conditions are valid for all linearly connected spaces, and involve the well-known euclidean conditions as a particular case.

Let x^i, $(i=1, 2, 3)$ be a system of coordinates in a given L_3, Γ^i_{jk} are prescribed coefficients of linear connection, $S_{jk}^{\cdot\cdot i} \equiv \Gamma^i_{[jk]} = \frac{1}{2}\left(\Gamma^i_{jk} - \Gamma^i_{kj}\right)$ is the torsion tensor and g_{ij} the fundamental tensor of the space, $\nabla_i g_{jk} = 0$, (where ∇_i denotes the covariant derivation with respect to the connection Γ^i_{jk}, $\nabla_i \Phi\ldots \equiv \Phi\ldots,_i$).

In an L_3 we can consider a set of equations

$$(1) \qquad \xi_{i,j} + \xi_{j,i} = 0.$$

The solutions present a set of pseudo-Killing vectors [5] in the space; in Riemannian spaces the solutions define the fundamental vectors of the group of isometric transformations (group of motions). Let us suppose that our L_3 admits a set of $r\,(>0)$ linearly independent (with respect to constant coefficients)

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[3] Numbers in square brackets refer to the list of references at the end of this note.

solutions $\xi_{(\lambda)i}$, $\lambda=1, 2, \ldots, r$. If V^i is a vector in L_3, we say that the *set of r scalars*

(2) $$M_{(\tau)} \overset{\text{def.}}{=} \xi_{(\tau)i} V^i$$

presents moments of the vector V^i[4].

If f_2 is an arbitrary closed surface in the material, bounding a volume f_3 in L_3, stress acting on a surface element df of f_2 normal to n_i is given by

(3) $$dp^i = \sigma^{ij} n_j \, df = \sigma^{ij} df_j,$$

where σ^{ij} is the stress tensor. Moments of stress on df, according to (2), will be given by scalars

(4) $$dm_{(\tau)} = \xi_{(\tau)i} dp^i = \xi_{(\tau)i} \sigma^{ij} df_j.$$

We shall assume the *couple stress* acting on the elementary surface area df to be a quantity of the same kind as moments of stress,

(5) $$d\tau_{(\tau)} = \tau^j_{(\tau)} df_j,$$

where $\tau^j_{(\tau)}$ are couple stress vectors, representing the couple stress per unit surface area (in the analogy with the stress tensor).

The total moment acting on the element df is now

(6) $$dM_{(\tau)} = dm_{(\tau)} + d\tau_{(\tau)} = [\xi_{(\tau)i} \sigma^{ij} + \tau^j_{(\tau)}] \, df_j.$$

Representing df_j by the skew-symmetric tensor of the 2. order,

$$df_j = \frac{1}{2} e_{jkl} \, df^{kl},$$

where e_{jkl} is the completely skew-symmetric tensor defined with respect to the fundamental tensor g_{ij} of L_3,

$$e_{jkl} = \begin{cases} +\sqrt{g} \\ -\sqrt{g} \\ 0. \end{cases} \qquad g = \text{Det}\,(g_{ij})$$

if jkl is an even, odd, or none permutation of 123, (6) can be written in the form

$$dM_{(\tau)} = \frac{1}{2} (\xi_{(\tau)i} \sigma^{ij} + \tau^j_{(\tau)}) \, e_{jkl} \, df^{kl}.$$

The total moment acting on f_2 is now given by

(7) $$\int_{f_2} dM_{(\tau)} = \frac{1}{2} \int_{f_2} (\xi_{(\tau)i} \sigma^{ij} + \tau^j_{(\tau)}) \, e_{jkl} \, df^{kl}.$$

[4] In [4] it is demonstrated that in euclidean spaces, and with respect to Cartesian orthogonal coordinates we have for „translational" moments $M_{(1)} = V^1$, $M_{(2)} = V^2$, $M_{(3)} = V^3$, and for „rotational": $M_{(4)} = x^2 V^3 - x^3 V^2$; $M_{(5)} = x^3 V^1 - x^1 V^3$; $M_{(6)} = x^1 V^2 - x^2 V^1$.

Since

$$T_{(\tau)kl} \equiv (\xi_{(\tau)i}\, \sigma^{ij} + \tau^{j}_{(\tau)})\, e_{jkl}$$

is a skew-symmetric tensor, Stokes' theorem can be applied to the right-hand side of (7) and we have

$$\int_{f_2} dM_{(\tau)} = \frac{1}{2} \int_{f_3} [\partial_j\, T_{(\tau)kl}]_{[jkl]}\, \partial f^{jkl},$$

where df^{jkl} is the volume element of f_3, and $[jkl]$ denotes „alternation" in the indices involved[5]. Now we have (cf. [6], p. 127)

(8) $$[\partial_j\, T_{(\tau)kl}]_{[jkl]} \equiv [\nabla_j\, T_{(\tau)kl} + 2\, S_{jl}^{\cdot\cdot m}\, T_{(\tau)km}]_{[jkl]}$$

In the absence of external moments and torques $dM_{(\tau)} = 0$ and the equilibrium conditions for moments of internal stress and couple stress read

(9) $$[\nabla_j\, T_{(\tau)kl} + 2 S_{jl}^{\cdot\cdot m}\, T_{(\tau)km}]_{[jkl]} = 0.$$

Substituting $T_{(\tau)kl}$ by its corresponding expression we obtain

$$\xi_{(\tau)i,m}\, \sigma^{im} + \tau^{m}_{(\tau),m} + 2 S_{mp}^{\cdot\cdot p}\, \tau^{m}_{(\tau)} + \xi_{(\tau)i}\, (\sigma^{im}{}_{,m} + 2 S_{mp}^{\cdot\cdot p}\, \sigma^{im}) = 0.$$

The equilibrium condition foı stress in L_3 reads [3]:

$$\sigma^{im}{}_{,m} + 2 S_{mp}^{\cdot\cdot p}\, \sigma^{im} = 0,$$

and we have

$$\xi_{(\tau)i,m}\, \sigma^{im} + \tau^{m}_{(\tau),m} + 2 S_{mp}^{\cdot\cdot p}\, \tau^{m}_{(\tau)} = 0.$$

Since $\xi_{(\tau)i}$ are pseudo—Killing vectors, we have from (1) $\xi_{(\tau)i,m} = -\xi_{(\tau)m,i}$, and the above equation can finally be written in the form

(10) $$\tau^{m}_{(\tau),m} + 2 S_{mp}^{\cdot\cdot p}\, \tau^{m}_{(\tau)} + \xi_{(\tau)i,j}\, \sigma^{[ij]} = 0.$$

$$\left[\sigma^{[ij]} \equiv \frac{1}{2}\, (\sigma^{ij} - \sigma^{ji})\right].$$

This expression reduces in euclidean spaces to the well-known form, if we consider only „rotational" moments (cf. footnote 4) and put

$$\tau^{ijm} \stackrel{\text{def.}}{=} e^{ij\tau}\, \tau^{m}_{(\tau)}.$$

Since in euclidean spaces $S_{mp}^{\cdot\cdot t} = 0$ and if $\xi_{(\tau)i}$ takes the values corresponding to the fundamental vectors (Killing vectors) of the rigid rotation, it follows

$$\tau^{ijm}{}_{,m} = 2\, \sigma^{[ij]}.$$

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[5] Sum of isomers of $\partial_j\, T_{(\tau)}{}^{kl}$ obtained by even permutations of the indices involved, minus the sum of odd permutations, devided by 3!

Acknowledgement. The authors wish to thank Professor Dr *E. Kröner* for making available to them the manuscript of the yet unpublished paper [2]. One of the authors *(R. S.)* is also indebted to Professor *Kröner* for many private communications and valuable discussions on the subject.

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