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CANONICAL TRANSFORMATIONS AND THE HAMILTON — JACOBI METHOD IN THE FIELD THEORY

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SUMMARY: Canonical transformations and Hamilton—Jacobi method are exposed on the ground of the theory of functionals and the generalized Pfaff—Bilimović method.

INTRODUCTION

Canonical transformations and Hamilton—Jacobi method are known to play a very important role in the analytical mechanics of the system of particles [1]. The development of these branches of mechanics began in the middle of the last century [2]. *W. Hamilton* [3] was the first to show that the problem of solving canonical, or so-called Hamilton equations can be reduced to the integration of a system of two first order partial differential equations with the same unknown function; this unknown function being the action with the undetermined upper limit of integration. The fact that the function to be determined had to satisfy two partial equations diminished to a great extent the significance of the method.

C. Jacobi [4] noted that one of the partial equations is not necessary, so that to any system of canonical equations corresponds but one first order partial differential equation, later named the Hamilton—Jacobi equation, the characteristics of which are integrals of the canonical system. The method for the solution of any given dynamical problem consists then in the determination of only one complete integral of the equation, i. e. an integral containing a number of arbitrary constants equal to the number of the independent variables of the problem. The so-called Jacobi system is formed then by differentiation of the found complete integral, and this system leads to the integral of the canonical equations through algebraic operations only. The problem of solving the canonical equations is thus reduced to the determination of a complete integral of the Hamilton—Jacobi partial equation and to the algebraic solution of the attached system.

S. Lie [5] studied this problem from the point of view of the general transformation theory, on the basis of the theory of groups. He introduced the notion of contact transformations to the theory of first order partial equations, and, as a special case, defined the so-called canonical transformations. Those are the transformations from the old to the new variables, possessing the property of leaving the form of the canonical equations invariant, and forming an Abelian group. If the integrals of the canonical equations are conceived as canonical transformations of the canonical variables from the values at the initial instant to those at any given instant, Lie's theory offers a new approach to the Hamilton—Jacobi method. The problem of determining such a canonical transformations which would yield a vanishing Hamiltonian, leads to the result that the generating function of the transformation has to satisfy exactly the Hamilton—Jacobi partial equation. The integrals of the canonical equations are determined then as above, and the Hamilton—Jacobi method is thus seen to be closely related to the theory of contact transformations.

Further extension of these results to the field theory is necessarily based on the theory of functionals, developped by *V. Volterra* [6, 7, 8]. This theory is of an increasing importance for various branches of theoretical physics operating with the notion of the field [9].

The very notion of the functional is a generalization of the notion of the function when the number of independent variables tends to infinity; likewise, the notions of functional derivative and differential are generalizations of those of partial derivative and total differential. A correspondance can, then, be established between ordinary mathematical analysis and the functional calculus on the ground of the so-called transition from discontinuum to continuum, if the ordinary functions are made to correspond to functionals, partial derivatives to functional ones, and total differentials to the functional.

Volterra himself was the first to apply the theory of the functionals to the mechanics of continuous media [10, 11]. He showed how one can obtain Hamilton equations, the Hamilton—Jacobi partial equation, and Jacobi theorem in this case from the corresponding equations for the system of particles, on the ground of the above mentioned principle of transition from discontinuum to continuum. But, this author just touched the problems, giving the equations without any analysis of the underlying problems from the mechanical point of view, so that the full meaning of his results and their mutual relations are not clearly seen.

A further step in the direction of application of the theory of functionals in the field theory is given in the work of *G. Moisil* [12]. He associated a set of coordinate functions to a given configuration of the continuous system, and obtained thus integro-differential equations similar to Lagrange equations. *G. Domokos* [13] recently gave a generalization of the Hamilton-Jacobi equation and the Jacobi theorem in covariant form, starting from the principle of variation of action. His results are, however, given without sufficient proof, some of the conclusions are not justified, and the example quoted is trivial.

We showed previously [14] that the theory of the functionals can serve as a basis to extend the Pfaff—Bilimović method to the field theory. Convenient definitions of the functional derivative and the differential for the functionals given in the form of definite integrals were introduced, and a generalization of Pfaffians and corresponding Pfaff equations was given, so that it was possible to show the way of a generalization of the Pfaff—Bilimović method.

This work will be devoted to a study of canonical transformations and of the Hamilton—Jacobi theory in the field theory, on the ground of the generalized Pfaff—Bilimović method. In the first part we purport to introduce the notion of the canonical transformations as integral functional transformations from the old to the new canonical variables, and to expose various types of the transformations. In the second part, an approach is made to the Hamilton—Jacobi method, by way of canonical transformations. It is, furthermore, shown how fundamental equations of this method can be obtained, and the results obtained are analyzed. A characteristic analogy with the mechanics of particle systems is noted.

1. CANONICAL TRANSFORMATIONS

Definition of the Canonical Transformation. — Consider certain integral transformations of the old field functions ψ_i and momentum densities π_i to new ones, $\overline{\psi}_i$ and $\overline{\pi}_i$, which are of the form:

$$(1.1)\qquad \begin{aligned} &\int \overline{\psi}_i \, dV = \int \mathfrak{F}_{1i}(\psi_i, \psi_{ij}, \pi_i, x_j, t)\, dV \\ &\int \overline{\pi}_i \, dV = \int \mathfrak{F}_{2i}(\psi_i, \psi_{ij}, \pi_i, x_j, t)\, dV, \qquad (i = 1, 2, \ldots, n); \end{aligned}$$

where we shall assume that the domain of integration is the entire volume V in wich the field functions are defined. Written in the form of functionals, these transformations are:

$$(1.2)\qquad \begin{aligned} &\int \overline{\psi}_i \, dV = F_{1i}[\psi_i, \pi_i, t] \\ &\int \overline{\pi}_i \, dV = F_{2i}[\psi_i, \pi_i, t], \qquad (i = 1, 2, \ldots, n). \end{aligned}$$

Under the assumption that these integral equations can be solved for volume integrals of the old variables, we have:

$$(1.3)\qquad \int \psi_i \, dV = G_{1i}[\overline{\psi}_i, \overline{\pi}_i, t]$$
$$\int \pi_i \, dV = G_{2i}[\overline{\psi}_i, \overline{\pi}_i, t], \qquad (i=1, 2, \ldots, n).$$

If a functional:

$$(1.4)\qquad \overline{H} = \int \overline{\mathfrak{H}} \, dV = \overline{H}[\overline{\psi}_i, \overline{\pi}_i, t]$$

exists, such that the differential equations of the field considered in the new variables preserve the form of the Hamilton equations, i. e.:

$$(1.5)\qquad \frac{d\overline{\pi}_i}{dt} = -\frac{\delta \overline{H}}{\delta \overline{\psi}_i}, \qquad \frac{d\overline{\psi}_i}{dt} = \frac{\delta \overline{H}}{\delta \overline{\pi}_i}; \qquad (i=1, 2, \ldots, n),$$

we shall say that (1.1) or (1.2) repıesent a *canonical transformation.*

Let us now examine the conditions necessary for a transformation to be canonical. In our previous work, it was shown how Hamilton equations can be obtained as Pfaff equations by the aid of the generalized Pfaff—Bilimović method, if the element of action, transformed to canonical form, is taken as the functional Pfaffian, i. e.:

$$\Phi = L\,dt = \int \left(\sum_{i=1}^{n} \pi_i \, d\psi_i - \mathfrak{H}\, dt \right) dV.$$

On the other hand, it was established that two finctional Pfaffians differring by a differential with respect to a parameter of any functional, are equivalent, i. e. yield the same functional Pfaff equations. The role of the parameter is played here by the time, so that it is possible to conclude that the condition looked for is:

$$(1.6)\qquad \int \left(\sum_{i=1}^{n} \pi_i \, d\psi_i - \mathfrak{H}\, dt \right) dV = \int \left(\sum_{i=1}^{n} \overline{\pi}_i \, d\overline{\psi}_i - \overline{\mathfrak{H}}\, dt \right) dV + dG,$$

where:

$$(1.7)\qquad G = \int \mathfrak{G} \, dV = G[\psi_i, \pi_i, \overline{\psi}_i, \overline{\pi}_i, t].$$

The equation (1.2), however, represents a system of $2n$ relations among the old and the new variables, so that only $2n$ among the arguments of the functional G are independent. This functional will be called *generating functional of the canonical transformation.*

Generating Functional of the Fundamental Type. — Let us take first the case in which G is a functional of the old and new field functions and time:

$$(1.8)\qquad G = G_1[\psi_i, \overline{\psi}_i, t].$$

In view of the formula for the differential with respect to parameter of a functional of the type considered, we have:

$$dG_1 = \int \left(\sum_{i=1}^{n} \frac{\delta G_1}{\delta \psi_i} d\psi_i + \sum_{i=1}^{n} \frac{\delta G_1}{\delta \overline{\psi}_i} d\overline{\psi}_i + \frac{\delta G_1}{\delta t} dt \right) dV \tag{1.9}$$

so that the condition (1.6) is:

$$\int \left(\sum_{i=1}^{n} \pi_i \, d\psi_i - \mathfrak{H} \, dt \right) dV = \int \left(\sum_{i=1}^{n} \overline{\pi}_i \, d\overline{\psi}_i - \overline{\mathfrak{H}} \, dt \right) dV +$$

$$+ \int \left(\sum_{i=1}^{n} \frac{\delta G_1}{\delta \psi_i} d\psi_i + \frac{\delta G_1}{\delta \overline{\psi}_i} d\overline{\psi}_i + \frac{\delta G_1}{\delta t} dt \right) dV.$$

In this case we have further:

$$\int \frac{\delta G_1}{\delta t} dV = \int \frac{\partial \mathfrak{G}_1}{\partial t} dV = \frac{\partial}{\partial t} \int \mathfrak{G}_1 \, dV$$

i. e.:

$$\int \frac{\delta G_1}{\delta t} dV = \frac{\partial G_1}{\partial t} \tag{1.10}$$

so that the above relation can be written in the form:

$$\sum_{i=1}^{n} \overline{d\psi_i} \int \pi_i \, dV - \sum_{i=1}^{n} \overline{d\overline{\psi}_i} \int \overline{\pi}_i \, dV + (\overline{H} - H) \, dt = \tag{1.11}$$

$$= \sum_{i=1}^{n} \overline{d\psi_i} \int \frac{\delta G_1}{\delta \psi_i} dV + \sum_{i=1}^{n} \overline{d\overline{\psi}_i} \int \frac{\delta G_1}{\delta \overline{\psi}_i} dV + \frac{\partial G_1}{\partial t} dt,$$

where the theorem of the mean values is used, the bars above the differentials denoting the mean values of these in the volume V. As the differentials $d\psi_i$, $d\pi_i$, and dt are mutually independent, the same will hold for their mean values $\overline{d\psi_i}$, $\overline{d\pi_i}$, and for dt; in the above equality, therefore, the coefficients of these magnitudes should be equal at both sides:

$$\int \pi_i \, dV = \int \frac{\delta G_1}{\delta \psi_i} dV, \quad \int \overline{\pi}_i \, dV = -\int \frac{\delta G_1}{\delta \overline{\psi}_i} dV, \qquad (i = 1, 2, \ldots, n)$$

$$\overline{H} = H + \frac{\partial G_1}{\partial t}. \tag{1.12}$$

These relations determine the canonical transformation with the generating functional of the type (1.8). Since they are of the form:

$$\int \pi_i \, dV = \int \varphi_{1i} \, (\psi_i, \psi_{ij}, \overline{\psi}_i, \overline{\psi}_{ij}, x_j, t) \, dV$$

$$\int \overline{\pi}_i \, dV = \int \varphi_{2i} \, (\psi_i, \psi_{ij}, \overline{\psi}_i, \overline{\psi}_{ij}, x_j, t) \, dV$$

or, functionally,

$$\int \pi_i\, dV = F_{1i}\,[\psi_i, \overline{\psi_i}, t]$$

(1.13)

$$\int \overline{\pi_i}\, dV = F_{2i}\,[\psi_i, \overline{\psi_i}, t],$$

they form a system of $2n$ integral equations with $\overline{\psi_i}$ and $\overline{\pi_i}$ as unknown functions. The first n equations of the system (1.12) are to be used to determine the new field functions, in terms of volume integrals:

(1.14) $$\int \overline{\psi_i}\, dV = G_{1i}\,[\psi_i, \pi_i, t]$$

and if the results are inserted into the second group of n equations of (1.12), the new momentum densities can be obtained in the form of volume integrals, too:

(1.15) $$\int \overline{\pi_i}\, dV = G_{2i}\,[\psi_i, \pi_i, t]$$

The canonical transformation (1.2) is thus determined, and the last equation of (1.12) then yields the new Hamiltonian $\overline{\text{H}}$.

Let us now formulate the condition for G_1 that the equations (1.12) can be solved. As the other half of these is already explicitly solved for momentum densities integrals, $\int \overline{\pi_i}\, dV$, it remains to establish only under what conditions the first half can be solved for $\int \overline{\psi_i}\, dV$. It is to be noted that if $\int \overline{\psi_i}\, dV$ can be determined, their differentials $d\int \overline{\psi_i}\, dV$ can also be found, and vice versa; on the other hand, since the differentiation with respect to parameter and integration over V are mutually independent operations, the mean value theorem yields:

$$d\int \overline{\psi_i}\, dV = \int d\overline{\psi_i}\, dV = \overline{\overline{d\psi_i}}\int dV$$

i. e.:

(1.16) $$d\int \overline{\psi_i}\, dV = V\cdot \overline{\overline{d\psi_i}}$$

so that the determination of $d\int \overline{\psi_i}\, dV$ can be reduced to determination of $\overline{\overline{d\psi_i}}$, in view of their mutual proportionality.

The first half of equations (1.12) then yields:

$$d\int \pi_i\, dV = d\int \frac{\delta G_1}{\delta \psi_i}\, dV = \int \Bigg\{ \sum_{j=1}^{n} \frac{\delta}{\delta \psi_j}\left(\int \frac{\delta G_1}{\delta \psi_i}\, dV \right) d\psi_j + \sum_{j=1}^{n} \frac{\delta}{\delta \overline{\psi_j}}\left(\int \frac{\delta G_1}{\delta \psi_i}\, dV \right) d\overline{\psi_j} + \frac{\delta}{\delta t}\left(\int \frac{\delta G_1}{\delta \psi_i}\, dV \right) dt \Bigg\}\, dV$$

which can be written in the form:

(1.17) $$\sum_{j=1}^{n} \overline{\overline{d\psi_j}} \int \frac{\delta}{\delta \overline{\psi_j}} \left(\int \frac{\delta G_1}{\delta \psi_i}\, dV \right) dV = \int \Bigg\{ d\pi_i - \sum_{j=1}^{n} \frac{\delta}{\delta \psi_j} \left(\int \frac{\delta G_1}{\delta \psi_i}\, dV \right) d\psi_j + \frac{\delta}{\delta t} \left(\int \frac{\delta G_1}{\delta \psi_i}\, dV \right) dt \Bigg\}\, dV.$$

This system is linear with respect to $d\overline{\psi}_j$, and can, therefore, be solved for these quantities if the determinant of their coefficients does not vanish:

$$(1.18)\qquad \left| \int \frac{\delta}{\delta\overline{\psi}_j}\left(\int \frac{\delta G_1}{\delta\psi_i}\,dV\right)dV \right| \neq 0;$$

and this is the condition looked for.

If $k=0$, i. e. if the field functions reduce to functions of time only, as in the case of a system of particles, the functionals will reduce to ordinary functions, functional derivatives to partial ones, and functional differentials to total differentials. In this case all volume integrals reduce to zero-fold integrals, i. e. to the integrands, and (1.2), (1.3), (1.6), (1.12), and (1.18) become wellknown relations of the mechanics of particle systems.

Other Types of Generating Functionals. — We shall now proceed to transform the condition (1.6), in order to obtain generating functionals depending on other arguments. Let us rewrite it first in the form:

$$(1.19)\qquad \sum_{i=1}^{n}\int \pi_i\,d\psi_i\,dV - \sum_{i=1}^{n}\int \overline{\pi}_i\,d\overline{\psi}_i\,dV + \int(\overline{\mathfrak{H}}-\mathfrak{H})\,dt\,dV = dG_1.$$

Since the differentiation with respect to parameter and integration with respect to space coordinates are mutually independent, we can interchange them to obtain:

$$d\int \overline{\pi}_i\overline{\psi}_i\,dV = \int d(\overline{\pi}_i\overline{\psi}_i)\,dV = \int(\overline{\pi}_i\,d\overline{\psi}_i + \overline{\psi}_i\,d\overline{\pi}_i)\,dV,$$

and, hence:

$$(1.20)\qquad \int \overline{\pi}_i\,d\overline{\psi}_i\,dV = d\int \overline{\pi}_i\overline{\psi}_i\,dV - \int \overline{\psi}_i\,d\overline{\pi}_i\,dV.$$

The above relation (1.19) can now be written as:

$$\sum_{i=1}^{n}\int \pi_i\,d\psi_i\,dV - \sum_{i=1}^{n} d\int \overline{\pi}_i\overline{\psi}_i\,dV + \sum_{i=1}^{n}\int \overline{\psi}_i\,d\overline{\pi}_i\,dV + \int(\overline{\mathfrak{H}}-\mathfrak{H})\,dt\,dV = dG_1$$

or:

$$(1.21)\qquad \int\left\{\sum_{i=1}^{n}\pi_i\,d\psi_i + \sum_{i=1}^{n}\overline{\psi}_i\,d\overline{\pi}_i + (\overline{\mathfrak{H}}-\mathfrak{H})\,dt\right\}dV = d\left(G_1 + \int\sum_{i=1}^{n}\overline{\pi}_i\overline{\psi}_i\,dV\right).$$

It is clearly seen that bracketed expression on the right can be treated as a functional of the variables ψ_i, $\overline{\pi}_i$, and t; we shall designate it by G_2:

$$(1.22)\qquad G_2 = G_1 + \int\sum_{i=1}^{n}\overline{\pi}_i\overline{\psi}_i\,dV = G_2\,[\psi_i, \overline{\pi}_i, t].$$

In view of the formula for the differential of the functional with respect to parameter, it follows:

$$d\left(G_1 + \int\sum_{i=1}^{n}\overline{\pi}_i\overline{\psi}_i\,dV\right) = dG_2 = \int\left(\sum_{i=1}^{n}\frac{\delta G_2}{\delta\psi_i}\,d\psi_i + \sum_{i=1}^{n}\frac{\delta G_2}{\delta\overline{\pi}_i}\,d\overline{\pi}_i + \frac{\delta G_2}{\delta t}\,dt\right)dV,$$

so that according to the theorem of the mean value and relation (1.10) we further have:

$$\sum_{i=1}^{n} \overline{d\overline{\psi}_i} \int \pi_i \, dV + \sum_{i=1}^{n} \overline{d\overline{\pi}_i} \int \overline{\psi}_i \, dV + (\overline{H} - H)\, dt = \tag{1.23}$$
$$= \sum_{i=1}^{n} \overline{d\overline{\psi}_i} \int \frac{\delta G_2}{\delta \overline{\psi}_i} dV + \sum_{i=1}^{n} \overline{d\overline{\pi}_i} \int \frac{\delta G_2}{\delta \overline{\pi}_i} dV + \frac{\partial G_2}{\partial t} dt.$$

After equating the coefficients of $\overline{d\overline{\psi}_i}$, $\overline{d\overline{\pi}_i}$ and dt on both sides, we obtain:

$$\int \pi_i \, dV = \int \frac{\delta G_2}{\delta \overline{\psi}_i} dV, \qquad \int \psi_i \, dV = \int \frac{\delta G_2}{\delta \overline{\pi}_i} dV \qquad (i = 1, 2, \dots n) \tag{1.24}$$
$$\overline{H} = H + \frac{\partial G_2}{\partial t}.$$

If the first term of (1.19) is transformed, the following relation will result:

$$-\sum_{i=1}^{n} \int \psi_i \, d\pi_i \, dV - \sum_{i=1}^{n} \int \overline{\pi}_i \, d\overline{\psi}_i \, dV + \int (\overline{\mathfrak{H}} - \mathfrak{H}) \, dt \, dV = \tag{1.25}$$
$$= d\left(G_1 - \int \sum_{i=1}^{n} \pi_i \psi_i \, dV\right),$$

so that upon introduction of a new functional:

$$G_3 = G_1 - \int \sum_{i=1}^{n} \pi_i \psi_i \, dV = G_3 [\pi_i, \overline{\psi}_i, t] \tag{1.26}$$

we obtain:

$$-\sum_{i=1}^{n} \overline{d\pi_i} \int \psi_i \, dV - \sum_{i=1}^{n} \overline{d\overline{\psi}_i} \int \overline{\pi}_i \, dV + (\overline{H} - H)\, dt = \tag{1.27}$$
$$= \sum_{i=1}^{n} \overline{d\pi_i} \int \frac{\delta G_3}{\delta \pi_i} dV + \sum_{i=1}^{n} \overline{d\overline{\psi}_i} \int \frac{\delta G_3}{\delta \overline{\psi}_i} dV + \frac{\partial G_3}{\partial t} dt,$$

and hence:

$$\int \psi_i \, dV = -\int \frac{\delta G_3}{\delta \pi_i} dV, \qquad \int \overline{\pi}_i \, dV = -\int \frac{\delta G_3}{\delta \overline{\psi}_i} \qquad (i = 1, 2, \dots, n) \tag{1.28}$$
$$\overline{H} = H + \frac{\partial G_3}{\partial t}.$$

Finally, if both first and second terms of (1.19) are transformed, one obtains:

$$-\sum_{i=1}^{n} \int \psi_i \, d\pi_i \, dV + \sum_{i=1}^{n} \int \overline{\psi}_i \, d\overline{\pi}_i \, dV + \int (\overline{\mathfrak{H}} - \mathfrak{H}) \, dt \, dV = \tag{1.29}$$
$$= d\left(G_1 + \int \sum_{i=1}^{n} \overline{\pi}_i \overline{\psi}_i \, dV - \int \sum_{i=1}^{n} \pi_i \psi_i \, dV\right),$$

and, if the new functional:

$$G_4 = G_1 + \int \sum_{i=1}^{n} \overline{\pi}_i \overline{\psi}_i \, dV - \int \sum_{i=1}^{n} \pi_i \psi_i \, dV = G_4 [\pi_i, \overline{\pi}_i, t] \tag{1.30}$$

is introduced, it results:

$$\begin{aligned} -\sum_{i=1}^{n} \overline{d\pi_i} \int \psi_i \, dV + \sum_{i=1}^{n} \overline{d\overline{\pi}_i} \int \overline{\psi}_i \, dV + (\overline{H} - H) \, dt = \\ = \sum_{i=1}^{n} \overline{d\pi_i} \int \frac{\delta G_4}{\delta \pi_i} dV + \sum_{i=1}^{n} \overline{d\overline{\pi}_i} \int \frac{\delta G_4}{\delta \overline{\pi}_i} dV + \frac{\partial G_4}{\partial t} dt \end{aligned} \tag{1.31}$$

and, hence:

$$\begin{aligned} \int \psi_i \, dV = -\int \frac{\delta G_4}{\delta \pi_i} dV, \quad \int \overline{\psi}_i \, dV = \int \frac{\delta G_4}{\delta \overline{\pi}_i} dV \qquad (i = 1, 2, \ldots, n) \\ \overline{H} = H + \frac{\partial G_4}{\partial t}. \end{aligned} \tag{1.32}$$

The established relations permit the determination of the canonical transformations in a manner analogous to that of equations (1.12).

2. HAMILTON — JACOBI METHOD

Hamilton — Jacobi Equation. — Consider now a canonical transformation of the type (1.2) for which the transformed Hamiltonian (1.4) vanishes, i. e.

$$\overline{H} = \int \overline{\mathfrak{H}} \, dV = 0; \tag{2.1}$$

assume, further, that the generating functional is of the type (1.22) and designate is by S:

$$G = G_2 = S [\psi_i, \overline{\pi}_i, t]. \tag{2.2}$$

According to the formula for the differential of the functional with respect to parameter, in view of (2.1) we have:

$$d\overline{H} = \int \left(\sum_{i=1}^{n} \frac{\delta \overline{H}}{\delta \overline{\psi}_i} d\overline{\psi}_i + \sum_{i=1}^{n} \frac{\delta \overline{H}}{\delta \overline{\pi}_i} d\overline{\pi}_i + \frac{\delta \overline{H}}{\delta t} dt \right) dV = 0$$

and, considering further (1.10), also:

$$\int \frac{\delta \overline{H}}{\delta t} dV = \frac{\partial \overline{H}}{\partial t} = 0,$$

so that the above equation can be written in the following form, using the theorem of the mean value:

$$\sum_{i=1}^{n} \overline{d\overline{\psi}_i} \int \frac{\delta \overline{H}}{\delta \overline{\psi}_i} dV + \sum_{i=1}^{n} \overline{d\overline{\pi}_i} \int \frac{\delta \overline{H}}{\delta \overline{\pi}_i} dV = 0. \tag{2.3}$$

Since the mean values $d\overline{\psi}_i$ and $d\overline{\pi}_i$ of the differentials are independent, this equality will hold only if the corresponding coefficients are zero; thus:

$$\int \frac{\delta \overline{H}}{\delta \overline{\psi}_i} dV = 0, \quad \int \frac{\delta \overline{H}}{\delta \overline{\pi}_i} dV = 0 \qquad (i = 1, 2, \ldots, n). \tag{2.4}$$

Hamilton equations then yield:

$$-\int \frac{d\overline{\pi}_i}{dt} dV = 0, \quad \int \frac{d\overline{\psi}_i}{dt} dV = 0,$$

or, since the differentiation with respect to time and integration over volume are independent and thus interchangeable:

$$\frac{d}{dt}\int \overline{\pi}_i \, dV = 0, \quad \frac{d}{dt}\int \overline{\psi}_i \, dV = 0. \tag{2.5}$$

These equations clearly show that the integrals involved should be equal to arbitrary constants:

$$\int \overline{\pi}_i \, dV = A_i, \quad \int \overline{\psi}_i \, dV = B_i, \qquad (i = 1, 2, \ldots, n), \tag{2.6}$$

and, since these arbitrary constants can be represented as volume integrals of arbitrary functions of the position variables, it is easily concluded that the new variables $\overline{\pi}_i$ and $\overline{\psi}_i$ are arbitrary functions of the position coordinates, but not of time; thus:

$$\overline{\pi}_i = \alpha_i(x_j), \quad \overline{\psi}_i = \beta_i(x_j), \qquad (i = 1, 2, \ldots, n), \tag{2.7}$$

α_i and β_i designating the arbitrary functions.

Since the generating functional of the canonical transformation considered was chosen to be of the type G_2, equations (1.24) and the condition (2.1) yield:

$$\int \pi_i \, dV = \int \frac{\delta G_2}{\delta \psi_i} dV, \quad \int \overline{\psi}_i \, dV = \int \frac{\delta G_2}{\delta \overline{\pi}_i} dV$$

$$\overline{H} = H + \frac{\partial G_2}{\partial t} = 0$$

which, in view of (2.2) and (2.7) can be written as:

$$\int \pi_i \, dV = \int \frac{\delta S}{\delta \psi_i} dV, \quad \int \beta_i \, dV = \int \frac{\delta S}{\delta \alpha_i} dV, \qquad (i = 1, 2, \ldots, n) \tag{2.8}$$

and:

$$\frac{\partial S}{\partial t} + H = 0. \tag{2.9}$$

In this last equality, H is a functional of the form:

$$H = \int \mathfrak{H} \, dV = H[\psi_i, \pi_i, t].$$

The first set of equations (2.8) shows then that π_i as an integrand can be replaced by $\frac{\delta S}{\delta \psi_i}$; it is in this sense that we write:

$$\pi_i = \frac{\delta S}{\delta \psi_i}, \qquad (i = 1, 2, \ldots, n), \tag{2.10}$$

and if this equation is used in the Hamiltonian density, it results:

$$H = H\left[\psi_i, \frac{\delta S}{\delta \psi_i}, t\right]$$

so that (2.9) becomes of the form:

$$\frac{\partial S}{\partial t} + H\left[\psi_i, \frac{\delta S}{\delta \psi_i}, t\right] = 0. \tag{2.11}$$

This functional differential equation is called the *Hamilton—Jacobi equation* of the field theory.

Jacobi Theorem. — We shall now show how a given problem of the field theory can be solved by the aid of the Hamilton—Jacobi equation. Let us assume that one solution of the Hamilton—Jacobi equation is determined, no matter how, and that it is of the integral form:

$$S = \int \mathfrak{S}\, (\psi_i, \psi_{ij}, \alpha_i, \alpha_{ij}, x_j, t)\, dV + S_0 \tag{2.12}$$

where α_i are arbitrary functions of the position coordinates, and S_0 an arbitrary constant. Written as a functional, this solution has the form:

$$S = S\,[\psi_i, \alpha_i, t] \tag{2.13}$$

and it is the so-called *complete integral* of the Hamilton—Jacobi equation considered.

In the case we are examining here, the new variables $\bar{\pi}_i$ and $\bar{\psi}_i$ are, in view of (2.7), arbitrary functions of the position coordinates, so that the arbitrary functions figurating in (2.13) can be taken to as new momentum densities $\bar{\pi}_i$:

$$\bar{\pi}_i = \alpha_i\,(x_j), \qquad (i = 1, 2, \ldots, n). \tag{2.14}$$

The new field functinos $\bar{\psi}_i$ will, then, be certain arbitrary functions of the position coordinates, too:

$$\bar{\psi}_i = \beta_i\,(x_j), \qquad (i = 1, 2, \ldots, n). \tag{2.15}$$

It should be noted that (2.8) contains only derivatives of the functional S, so that the arbitrary constant in (2.13) is not essential and can, therefore, be taken as zero. Relations (2.8) then yield:

$$\begin{aligned} \int \pi_i\, dV &= \int \frac{\delta S\,[\psi_i, \alpha_i, t]}{\delta \psi_i}\, dV \\ \int \beta_i\, dV &= \int \frac{\delta S\,[\psi_i, \alpha_i, t]}{\delta \alpha_i}\, dV \qquad (i = 1, 2, \ldots, n). \end{aligned} \tag{2.16}$$

This system of equations can be solved in a manner analogous to that applied in solving (1.12). They are of the form:

$$\int \pi_i \, dV = \int \varphi_{i1} (\psi_i, \psi_{ij}, \alpha_i, \alpha_{ij}, x_j, t) \, dV$$

$$\int \beta_i \, dV = \int \varphi_{i2} (\psi_i, \psi_{ij}, \alpha_i, \alpha_{ij}, x_j, t) \, dV$$

or, if the appropriate functional notation is introduced:

$$\begin{aligned} &\int \pi_i \, dV = \Phi_{1i} [\psi_i, \alpha_i, t] \\ &\int \beta_i \, dV = \Phi_{2i} [\psi_i, \alpha_i, t], \quad (i = 1, 2, \ldots, n) \end{aligned} \tag{2.17}$$

and represent, thus, a system of $2n$ integral equations with ψ_i and π_i as the unknown functions. From the second half of these equations, the field functions ψ_i can be determined in the form of volume integrals:

$$\int \psi_i \, dV = F_{1i} [\alpha_i, \beta_i, t], \qquad (i = 1, 2, \ldots, n), \tag{2.18}$$

and inserting these expressions into the first half of the system considered, momentum densities are likewise determined:

$$\int \pi_i \, dV = F_{2i} [\alpha_i, \beta_i, t], \qquad (i = 1, 2, \ldots, n). \tag{2.19}$$

Let us examine now the condition necessary for the system to be solvable. Since S is a functional of type G_2, differring from G_1 only by its dependence on $\bar{\pi}_i$, i. e. on $\bar{\alpha}_i$, instead of $\bar{\psi}_i$, by analogy with (1.18) we obtain, substituting $\bar{\psi}_j$ by $\bar{\alpha}_j$:

$$\left| \int \frac{\delta}{\delta \alpha_j} \left(\int \frac{\delta S}{\delta \psi_i} \, dV \right) dV \right| \neq 0. \tag{2.20}$$

We have thus seen that the solution of the Hamilton equations can be reduced to the determination of a complete integral of the Hamilton—Jacobi equation, so that the following statement can be formulated:

If a complete integral of the Hamilton—Jacobi equation is found in the form (2.13), *on condition* (2.20), *the solution of the equations* (2.16) *will yield the solution of the problem considered in the form* (2.18) *and* (2.19).

This is the *generalized Jacobi theorem.*

To demonstrate the meaning of the functional S, we shall form its differential with respect to time as parameter:

$$dS = \int \left(\sum_{i=1}^{n} \frac{\delta S}{\delta \psi_i} \, d\psi_i + \sum_{i=1}^{n} \frac{\delta S}{\delta \alpha_i} \, d\alpha_i + \frac{\delta S}{\delta t} \, dt \right) dV,$$

which, since in view of (1.10) we have:

$$\int \frac{\delta S}{\delta t} \, dV = \frac{\partial S}{\partial t},$$

yields:

$$dS = \sum_{i=1}^{n} \overline{d\psi_i} \int \frac{\delta S}{\delta \psi_i} dV + \sum_{i=1}^{n} \overline{d\alpha_i} \int \frac{\delta S}{\delta \alpha_i} dV + \frac{\partial S}{\partial t} dt. \tag{2.21}$$

According to (2.16), (2,11) and (2.6), we further have:

$$dS = \sum_{i=1}^{n} \overline{d\psi_i} \int \pi_i \, dV + \sum_{i=1}^{n} \overline{d\alpha_i} \int \beta_i \, dV - H \, dt = \int \left(\sum_{i=1}^{n} \pi_i \, d\psi_i - \mathfrak{H} \, dt \right) dV + \sum_{i=1}^{n} B_i \overline{d\alpha_i}.$$

The first term on the right is easily identified as the element of action transformed to the canonical form. Since the mean value of the differential is equal to the differential of the mean value, the above expression can be further transformed:

$$dS = L \, dt + \sum_{i=1}^{n} B_i \, d\overline{\alpha_i}, \tag{2.22}$$

and, upon integration with respect to time from t_0 to t:

$$S = \int_{t_0}^{t} L \, dt + \sum_{i=1}^{n} B_i \left[(\overline{\alpha_i})_t - (\overline{\alpha_i})_{t_0} \right].$$

The arbitrary functions being independent of time, their mean values should be equal at the two instants:

$$(\overline{\alpha_i})_t = (\overline{\alpha_i})_{t_0}$$

so that we have:

$$S = \int_{t_0}^{t} L \, dt. \tag{2.23}$$

Hence, the functional S is seen to coincide with the action with the variable upper limit of integration.

If $k = 0$, functionals reduce to ordinary functions, functional derivatives and differentials become ordinary partial derivatives and total differentials, and all integrals considered reduce to zero-fold ones, i. e. to the very integrands. In this case, the relations (2.1), (2.6), (2.11), (2.13), (2.16), (2.20) and (2.23) reduce to the well-known equations of the mechanics of the system of particles.

Case of Conservative Fields. — Consider now the case of a conservative field, where the Hamiltonian does not depend on time explicitly, and, under very general conditions, represents the energy of the field:

$$H = H[\psi_i, \pi_i] = E. \tag{2.24}$$

In this case, we shall look for a solution of the Hamilton—Jacobi equation in the form:

$$S = -A_1 t + S_1, \tag{2.25}$$

where:

$$A_1 = \int \alpha_1 \, dV, \quad S_1 = S_1[\psi_i, \alpha_i]. \tag{2.26}$$

Since:

$$\frac{\partial S}{\partial t} = -A_1, \quad \frac{\delta S}{\delta \psi_i} = \frac{\delta S_1}{\delta \psi_i},$$

inserting (2.25) into the Hamilton—Jacobi equation (2.11) we have:

$$-A_1+H\left[\psi_i, \frac{\delta S_1}{\delta \psi_i}\right]=0,$$

or

$$H\left[\psi_i, \frac{\delta S_1}{\delta \psi_i}\right]=A_1. \tag{2.27}$$

The constant A_1, according to (2.24) is the energy of the field, and the obtained equation is the Hamilton—Jacobi equation for the conservative fields.

Let us now assume that one of its solutions is determined in the form:

$$S_1=S_1[\psi_i, \alpha_i]+S_0. \tag{2.28}$$

Since, according to (2.25) and (2.26):

$$\frac{\delta S}{\delta \alpha_i}=-\frac{\delta A_1}{\delta \alpha_i}t+\frac{\delta S_1}{\delta \alpha_i}=-\frac{\partial \alpha_1}{\partial \alpha_i}t+\frac{\delta S_1}{\delta \alpha_i}=\frac{\delta S_1}{\delta \alpha_i}-\delta_{1i}\,t,$$

where δ_{1i} denotes Kronecker symbol, equal to unity for $i=1$, and zero otherwise, relations (2.16) become:

$$\begin{aligned}\int \pi_i\, dV&=\int \frac{\delta S_1[\psi_i, \alpha_i]}{\delta \psi_i}\, dV\\ \int \beta_i\, dV&=\int \left\{\frac{\delta S_1[\psi_i, \alpha_i]}{\delta \alpha_i}-\delta_{1i}\,t\right\} dV \qquad (i=1, 2, \ldots, n).\end{aligned} \tag{2.29}$$

This system is solved in a manner similar to that applied in (2.16), and the solution is of the form:

$$\begin{aligned}\int \psi_i\, dV&=F_1'[\alpha_i, \beta_i, t]\\ \int \pi_i\, dV&=F_2'[\alpha_i, \beta_i, t] \qquad (i=1, 2, \ldots, n).\end{aligned} \tag{2.30}$$

Method of Separation of Variables. — Consider again a conservative field, and assume that its Hamiltonian can be represented as a sum of terms, each depending on one single field function and the corresponding momentum density only. If the underlined variables are used to express the i-th member of the set of variables only, the above assumption can be written in the form:

$$H=\sum_{i=1}^{n} H_i[\underline{\psi_i}, \underline{\pi_i}], \tag{2.31}$$

so that equation (2. 27) becomes:

$$\sum_{i=1}^{n} H_i\left[\underline{\psi_i}, \underline{\frac{\delta S_1}{\delta \psi_i}}\right]=A_1 \tag{2.32}$$

We shall attempt now to find a solution of this equation in the form:

$$S_1=\sum_{i=1}^{n} S_{1i}[\underline{\psi_i}, \underline{\alpha_i}], \tag{2.33}$$

since:

$$\frac{\delta S_1}{\delta \psi_i} = \frac{\delta S_{1i}}{\delta \psi_i},$$

the equation (2.32) yields:

$$\sum_{i=1}^{n} H_i \left[\underline{\psi_i}, \frac{\delta S_{1i}}{\underline{\delta \psi_i}} \right] = A_1. \tag{2.34}$$

The left side of this equation is a sum of terms each depending on one single field function and the corresponding momentum density only, so that the equation will be satisfied if each term equals a constant; thus:

$$H_i \left[\underline{\psi_i}, \frac{\delta S_{1i}}{\underline{\delta \psi_i}} \right] = A_{1i}, \qquad (i = 1, 2, \ldots, n), \tag{2.35}$$

with the condition:

$$\sum_{i=1}^{n} A_{1i} = A_1. \tag{2.36}$$

Hence, the assumption (2.31) leads to a decomposition of the equation (2.27) to n mutually independent equations (2.35), each containing only one unknown functional S_{1i} and one corresponding independent variable ψ_i. When the solutions of these n equations are determined, the solution of (2.27) is given by (2.33).

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