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THE CARTESIAN MULTIPLICATION AND THE CELLULARITY NUMBER

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1. Introduction

There are many questions in connexion with the cartesian multiplications of sets, structures etc. In particular, the question is to find how some property of the cartesian product is induced by the analogous property of the factors. Some classical facts show that big differences may occur between the factors and the product. E. g. the problem of measure on the line R and in the square (plane) R^2 are of a different kind then the problem of the measure in the space R^3 or in spaces R^n for $n>2$. The problem whether the cardinality $k\,E^2$ of infinite sets E equals $k\,E$ is equivalent to the choice axiom.

In this article we shall examine a particular number — the cellularity number $c\,X\,(=\mathrm{cel}\,X)$ where X is any family of sets, any topological space or structure in order to see how the cellularity of product depends on the cellularity of the factors. At the same time, we shall become aware how the complete answer to the problem is connected with the tree hypothesis, and with the general continuum hypothesis. At this opportunity it is interesting to observe that the chain $\times$ antichain hypothesis holds for every square or hypersquare of every tree or ramified set.

In the present first part of the paper main results are contained in theorems 3.4; 3.7; 4.10; 5.4; 5.6; 5.8 and 6.4.

2. Cellularity of a system of sets

2.1. For any system S of sets we define the cellularity $c=c\,S=cel\,S$ of S by the relation

$$c\,S=\sup_{\Phi} k\,\Phi, \tag{1}$$

Φ consisting of pairwise disjoint sets belonging to S; $k\,\Phi$ means the cardinality of Φ.

2.2. For any space E the cellularity $c\,E$ of E is defined as the cellular number of the family of all open sets of the space E.

2.3. For any totally ordered set O the cellularity $c\,O$ of O is defined as the cellular number of the system of sets of the form

$$O\,[x,\,.\,)=\{y;\ x\leqslant y;\ \{x,y\}\subseteq O\}$$

$$O\,(.\,,x]=\{y;\ y\leqslant x;\ \{x,y\}\subseteq O\}$$

$$O\,(x,y)=\{z;\ x<z<y \text{ or } x<z<y;\ \{x,y,z\}\subseteq O\}.$$

3. Cartesian multiplication of sets and of families of sets, respectively

3.1 Definition. Let I be any set of >1 elements (the members of I shall serve as indices) ; for any $i\in I$ let X_i be a nonvoid set; *the cartesian product of the sets* X_i *is the set* $Y=\prod\limits_{i\in I} X_i$ *consisting of all the single-valued functions* f *on* I *such that for every* $i\in I$ *one has* $f(i)\in X_i$; i. e.

$$Y=\prod_{i\in I} X_i=\{f;\ f: I\to \bigcup_{i\in I} X_i,\ f(i)\in X_i\};$$

for $f, g\in Y$ one defines $f=g\Leftrightarrow f(i)=g(i)\ (i\in I)$.

In particular, for any ordinal number α one defines the hypercube X^α as the set of all the α-sequences of members in X; X^0 means the empty set.

3.2. Let I be a non void set and F a mapping on I such that, for every $i\in I$, F_i be a nonempty family of non void sets; the cartesian product of the families F_i is the family $F^{\cdot I}$ of the products $\prod\limits_{i\in I} X_i$ where $X_i\in F_i\ (i\in I)$. In particular, for any ordinal α and any family F of sets one defines $F^{\cdot I}=F^{\cdot I\alpha}=\{\prod\limits_i X_i;\ X_i\in F,\ i<\alpha;\ I_\alpha$ means the set of all the ordinal numbers $<\alpha\}$.

$F^{\cdot 2}$ consists of all the products $X_0\times X_1$ where X_0, X_1 run independently through F. One proves readily the following lemma.

3. Lemma. For every disjoint (antidisjoint) family F of sets one has $(c\,F)^{k\,I\alpha}=c\,(F^{\cdot I\alpha})$.

3.3. Definition. A family of sets is called *disjoint (antidisjoint)*, provided its members are pairwise disjoint (nondisjoint).

3.4. Main theorem. *(I) For any family F of sets and any natural number n the relation* $c\,F\geqslant\aleph_0$ *implies*

$$(1)\qquad (c\,F)^n\leqslant c\,(F^{\cdot n})\leqslant 2^{cF};$$

(II) For any ordinal number α there exists a system F of sets such that

$$(2)\qquad c\,F_\alpha=\aleph_\alpha,\quad (c\,F^{\cdot 2})=2^{\aleph_\alpha}$$

Therefore the evaluation in (1) *is a best one.*

3.5. Proof of the theorem 3.4. (I). 1. The first relation (1) is obvious because for every disjoint system d of sets in F we have the system $d^{\cdot r}$ in $F^{\cdot r}$ that is disjoint and of a cardinality $\geqslant kd$. Therefore, we have still to prove the second relation in (1). The proof will be carried out by induction relative to n.

First of all the relation (1) holds for $n=2$. The proof of this fact is quite characteristic. We have to prove that every disjoint system D of sets in $F^{\cdot I_2}$ is of a cardinality $\leqslant 2^{cF}$.

2. Now, let D be any disjoint system of the family $F^{\cdot I_2}$; this means that $X, Y \in D \Rightarrow X \cap Y = \varnothing$ or $X = Y$.

Now, $X = X_0 \times X_1$, $Y = Y_0 \times Y_1$, X_0, X_1, Y_0, Y_1 being elements of F. The relation

$$(X_0 \times X_1) \cap (Y_0 \times Y_1) = \varnothing$$

is equivalent to the disjunction

$$X_0 \cap Y_0 = \varnothing \vee X_1 \cap Y_1 = \varnothing .$$

Let $(D; \rho)$ be the binary graph supported by D and where the relation $X \rho Y$ means that $X_0 \cap Y_0 = \varnothing$ holds. Thus if C is a ρ-chain in $(D; \rho)$, then $C_0 = \{X_0;\ X \in C\}$ is a disjointed system of F and therefore $k C_0 \leqslant cF$. If $\bar{C}$ is an antichain in (D, ρ), then $\{X, Y\} \in \bar{C}$ implies the negation $X \rho' Y$ of $X \rho Y$ i. e. that $X_0 \cap Y_0 \neq \varnothing$ and consequently $X_1 \cap Y_1 = \varnothing$; this means that again $\bar{C}_1 = \{X_1;\ X \in \bar{C}\}$ is a disjointed system in F. Consequently, every chain as well as every antichain of (D, ρ) is $\leqslant cF$. In virtue of our graph theorem we have

$k D \leqslant 2^{cF}$ (cf. [4]. 3 Theorem 0.1 p. 82 and [4]. 4 Theorem 6.2.2).

This holding for every disjointed system D in $F^{\cdot 2}$ one has

$$\sup_D k D \leqslant 2^{cF} \text{ i. e. } cF^{\cdot 2} \leqslant 2^{cF}.$$

Consequently, the theorem holds for $n=2$.

3. Now, suppose that r be any natural number > 2 and the relation (1) holds for any natural number $n < r$; let us prove that (1) holds for $n = r$ too. Now, let D be any disjoint system in $F^{\cdot n}$; then for $X = (X_0 \times X_1 \times \ldots \times X_{r-1}) \in F$ and $Y = (Y_0 \times Y_1 \times \ldots \times Y_{r-1}) \in F$ the disjonction $X \cap Y = \varnothing$ means

$$(3) \qquad (X_0 \times X_1 \times \ldots \times X_{r-2}) \cap (Y_0 \times Y_1 \times \ldots \times Y_{r-2}) = \varnothing$$

or

$$X_{r-1} \cap Y_{r-1} = \varnothing .$$

With respect to the relation (3) the subset D of $F^{\cdot r}$ is a binary graph; by an argument like in 2 one proves that the induction hypothesis implies that every chain of this graph is $\leqslant 2^{cF}$ and that every antichain of the graph is $\leqslant cF$; in virtue of the graph theorem we infer that

$k D \leqslant (2^{cD})^{cD} = 2^{cD}$; this holding for every D, the operator sup yields (1). And this was to be shown.

3.6. Proof of the theorem 3.3. (II). Let α be any ordinal number and let

$$M = Q(\omega_\alpha)$$

be the system of all the ω_α — sequences of rational numbers ordered by the principle of the first differences: for any 2 different such sequences a, b let $i = i(a, b)$ be the ordinal such $a_{i'} = b_{i'}$ for every ordinal $i' < i$ and $a_i \neq b_i$; we put $a < b$ if and only if $a_i < b_i$.

3.6.1. Lemma. $kQ(\omega_\alpha)=\aleph_0^{\aleph_\alpha}\,(=2^{\aleph_\alpha})$.

3.6.2. M is a chain with respect to the relation $<$ and every interval of M has $2^{\aleph_\alpha}$ points.

In fact let $a=(a_\nu)_\nu$, $b=(b_\nu)_\nu$ be 2 distinct elements of M; hence $i=i(a,b)<\omega_\alpha$ and either $a_i<b_i$ or $a_i>b_i$; if then c is any element of M such that $c_i\in Q(a_i,b_i)$, $i(a,c)=i(b,c)$, one has $c\in M(a,b)$; in particular, the ω_α-sequence $c_{i+1},\ldots,c_{i+\omega'\alpha}$, might be any ω_α-sequence of rational numbers.

3.6.3. Lemma. *Any increasing (decreasing) sequence in M is of a cardinality* $\leqslant\aleph_\alpha$.

First of all the set M contains a ω_α-sequence as well as an ω^*_α-sequence; such are e. g. the sequences:

$$a^\zeta=\{1\}_\zeta\dot{+}\{0\}_{-\zeta+\omega_\alpha}\quad(\zeta<\omega_\alpha)$$

$$b^\zeta=\{0\}_\zeta\dot{+}\{1\}_{-\zeta+\omega_\alpha}\quad(\zeta<\omega_\alpha).$$

Further let us suppose that $(M,<)$ contains a well-ordered subset W of cardinality $>\aleph_\alpha$. In particular we might suppose that the type of W be $\omega_{\alpha+1}$. Now, every member x of W is a ω_α-sequence (x_ζ) with $x_\zeta\in Q$; for any pair x,y of distinct members of M let $i(x,y)$ be the first ordinal ν such that $x_\nu\neq y_\nu$. The ordinal $i(x,y)$ is like *a proximity degree (or dual distance)* between x, y and one proves readily that

(1) $$x<y<z\Rightarrow i(x,z)=\inf\{i(x,y),\ i(y,z)\}.$$

This relation is like triangular relation.

Consequently, for every member $x\in W$ we have the non decreasing monotone sequence

(2) $$i(x,y),\ (y\in W(x,.))$$

of ordinal numbers $<\omega_\alpha$; let $g(x)$ be the first $y>x$ in W such that

$i(x,gx)$ equals the infimum of the numbers (2).

In other words

(3) $$i(x,g(x))=\inf\ i(x,y),\quad(y\in W(x,.)).$$

The relations (1) and (3) yield the following relation

(4) $$i(x,y)=i(x,g(x)),\ (y\in W(g(x),.)).$$

Geometrically, the relation (4) means that the terminating interval $W(gx,.)$ of W is located on the „sphere“ $S(x,r^*)$, the center and the dual radius r^* of which are x and $r^*=i(x,gx)$ respectively; at the same time, gx is the first point of $W(x,.)$ located on this sphere.

Now by induction procedure we shall prove that the space $(W;i)$ (or ordered set $(W,<)$) would contain a subset $K=(k_0<k_1<\ldots)$ of cardinality $\aleph_{\alpha+1}$ of points with a *constant mutual proximity* δ or there would be a decreasing sequence of cardinality $\aleph_{\alpha+1}$ of „spheres“ (or terminating intervals of $(W,<)$) having no point in commun. None of these possibilities might occur in the present case. For the last eventuality the thing is obvious; as to the first eventuality, the set K would be a well-ordered subset of $(W, <)$ and for every 2-point-set $\{x, y\} \subset K$ one would have $i(x, y) = \delta$; the set K of all the δ^{th} coordinates x_δ of members x of K would be a subset of $(W, <)$ isomorphic with $(K, <)$—absurdity.

To start with, let $k_0 = W_0$; put $k_1 = W_{i(k_0, gk_0)}$ (cf. (3)); suppose that ν be an ordinal $< \omega_{\alpha+1}$ and that the decreasing „spheres" $S(k_\zeta; \overset{*}{r}_\zeta) = W(g\,k_\zeta, .)$ $(\xi < \nu)$ with $\overset{*}{r}_\zeta = i(k_\zeta, gk_\zeta)$ are defined; we put $k_\nu = W_{i(k_{\nu-1}, gk_{\nu-1})}$ or $k_\nu = \sup_{\zeta<\nu} k_\zeta$, according as $\nu - 1$ is limit or non limit ordinal. The construction of k_ν is well determined for every $\nu < \omega_{\alpha+1}$ and one sees by induction argument that really

(5) $\quad S(k_\nu, \overset{*}{r}_\nu) = W(gk, .)$ for every $\nu < \omega_{\alpha+1}$; in other words

(6) $\quad i(k_\nu, y) = \overset{*}{r}_\nu = i(k_\nu, gk_\nu)$ for every $y \in W(g\,k_\nu, .)$.

The function $\nu \to \overset{*}{r}_\nu$ is a monotone non decreasing function of $I\,\omega_{\alpha+1}$ into $I\,\omega_\alpha$. Let r^* be the supremum of the ordinals $\overset{*}{r}_\nu$. One has

(7) $$r^* \leqslant \omega_\alpha .$$

Now, the relat on $r^* = \omega_\alpha$ would imply that some ω_α-sequence of segments $W(g\,k^\zeta, .)_\zeta$ would have a void intersection (take e. g. k^ζ as the first k_ν satisfying $i(k_\nu, gk_\nu) = \overset{*}{r}_\zeta$); in other words the ω_α-sequence gk^ζ would be cofinal with the $\omega_{\alpha+1}$-sequence W—absurdity.

The relation

(8) $$r^* < \omega_\alpha$$

does not hold neither. Namely, if the number r^* is isolated, there would be $r^* = r^*_\mu = i(k_\mu, gk_\mu)$ for a $\mu < \omega_{\alpha+1}$; if r^* is non isolated, then for some strictly increasing sequence $r^*_{n_\zeta} = i(k^\zeta, g(k^\zeta))$ of cardinality $\leqslant \aleph_\alpha$ there would be $r^* = \sup r^*_{n_\zeta}$; in either case one concludes that $r^* = r^*_\nu$ for every ν of the final section $S = K(z, .)$, where $z = k_\mu$ or $z = \sup k^\zeta$. According to (6) this means that $i(k_\nu, y) = r^* = i(k_\nu, gk_\nu)$ for every $y \in W[z, .)$. Therefore by (3) we infer that all distinct points in $W[z, .)$ have a same mutual proximity — the number r^*. This fact implies that the set $W[z, .)$ we defined above is a subset of Q isomorphic to $W[z, .)$ and W — absurdity.[1]

3.6.4. *A partial order associated to the linear order* $(M; <)$.

Let $x \to ux$ $(x \in M)$ be a normal well-ordering uM of M i. e. such that uM be nonequivalent to any of its proper initial portions; in other words let $u = ux$ be a one-to-one mapping of M onto the segment of ordinal numbers corresponding to an initial ordinal ω_β. Let then the partial order $\prec$ in M be defined as superposition of the orders $<$ and u:

$$a \preccurlyeq b \text{ means } a \leqslant b \text{ and } ua \leqslant ub.$$

1. *Every chain C in $(M \prec)$ is of a cardinality* $\leqslant \aleph_\alpha$.

In fact C is a well-ordered subset in $(M; <)$ and in virtue of 3.6.3 C is $\leqslant \aleph_\alpha$.

[1] The foregoing proof of Lemma 3.6.1 represents a space-theoretical wording (using abstract distance or abstract proximity) of the theorem **XIV** in Hausdorff [1].

2. *Every antichain A in $(M; \prec)$ is of a cardinality* $\leqslant \aleph_\alpha$.

As a matter of fact, A is a decreasing sequence in $(M; <)$ and in virtue of 3.6.3 A is $\leqslant \aleph_\alpha$.

3.6.5. Let $x \in M$ and (cf. Sierpiński [5])

$$E_x = \{\{x, x'\};\ x' \in M,\ x' \prec x \text{ or } x \prec x'\}.$$

The mapping $x \rightarrow E_x$ is biunique.

For if $y \in M$ and e. g. $x < y$ let then $z \in M(x, y)$ and $uz > \sup\{ux, uy;\}$ one has $x \prec z$ and thus $\{x, z\} \in E_x$; on the contrary $\{x, z\}$ non $\in E_y$, because y non $\in \{x\, z\}$.

3.6.6. Let $F = \{E_x;\ x \in M\}$.

Then $kF = kM = 2^{\aleph_\alpha}$.

3.6.7. We consider the graph $(F; D)$, D being the disjunction relation. *Every antichain as well as every chain of the graph $(F; D)$ is* $\leqslant \aleph_\alpha$.

In fact, let A be an antichain in $(F; D)$; let E_x, E_y be two distinct elements of A; then $\{x, y\}_{\neq} \subset M$ and $E_x \cap E_y \neq \varnothing$; let $\{x, x'\} = \{y, y'\}$ be an element of E_x and E_y; then $x' = y$, $y' = x$; consequently, the points x, y are $\prec$—comparable in M; and vice versa, if x, y are 2 distinct $\prec$—comparable points of $(M; \prec)$, then $E_x \cap E_y \neq \varnothing$. If $E_x \cap E_y = \varnothing$, then x, y are not $\prec$—comparable:

$$x \not\equiv y\ (\cdot \prec) \Leftrightarrow E_x \cap E_y = \varnothing$$

$$x \equiv y\ (\cdot \prec) \Leftrightarrow E_x \cap E_y \neq \varnothing.$$

Consequently, to every $\prec$—chain C in $(M; \prec)$ corresponds the I—chain consisting of the elements $E_x\,(x \in C)$; to every $\prec$—antichain A corresponds the disjointed system $E_x\,(x \in A)$.

As a consequence of 3.6.3. one has therefore 3.6.7.

3.6.8. The system G of sets.

For any $x \in M$ let $G_x = \{\{x, y\}_{\neq};\ y$ is $\prec$— incomparable to $x\}$ i. e. $(x < y) \wedge (ux > uy) \vee (x > y \wedge (ux < uy)$. Let $G = \{G_x;\ x \in M\}$.

1. Every chain and every antichain in $(G; D)$ is $\leqslant \aleph_\alpha$. Again, $x \not\equiv y\,(\cdot \prec) \Leftrightarrow G_x \cap G_y \neq \varnothing$ i. e. x comp. $y \Leftrightarrow G_x \cap Gy = \varnothing$.

3.6.9. $F \cdot \cap \cdot G = \varnothing$ i. e. $x \in F \wedge y \in G \Rightarrow x \cap y = \varnothing$.

3.6.10. *Family H. Let $H = F \cup G$; the family $(H; D)$ is the required family: every D-chain and every D-antichain is* $\leqslant \aleph_\alpha$.

Now $H^{\cdot I 2}$ contains a disjointed system of kM elements because the sets

$$H_i = E_i \times G_i\,(i \in M)$$

are pairwise disjoint. As a matter of fact, let $x \neq y$ and $x, y \in M$; then either x, y are comparable or incomparable in M; if x, y are comparable, then G_x, G_y are disjoint and so are the sets H_x, H_y; if x, y are incomparable, the sets E_x, E_y are disjoint and so are also the sets H_x, H_y. The theorem 3. 3. (II) is proved.

3.7. Theorem *For every family F of sets and every ordinal number α we have:* (1) $cF = cF^{\cdot I 2}$ *provided* $cF = 1$

(2) $(cF)^{k\alpha} \leqslant cF^{\cdot I\alpha} \leqslant kF^{k\alpha}$; *if* $cF > 1$, *then for some ordinal* α_0 *of cardinality* $\leqslant kF$ *we have*

(3) $(cF)^{k\alpha} = cF^{\cdot I\alpha}$ *for every ordinal* $\alpha \geqslant \alpha_0$.

The relation (1) is obvious; the first relation in (2) is a consequence of the fact that the cartesian product of a disjoint system of sets is again a disjoint system of sets. The second relation in (2) is obvious because the cellularity of any family of sets is less than or equals to the cardinality of the same family; on the other hand, the cardinality of $F^{\cdot I\alpha}$ equals $(kF)^{k\alpha}$. Therefore, the relations (2) hold. Finally, if $kF \leqslant k\alpha$, then $kF^{k\alpha} = 2^{k\alpha}$, and therefore according to (2) we have $cF^{I\alpha} \leqslant 2^{k\alpha}$; this relation joint with the relation $2^{k\alpha} \leqslant (cF)^{k\alpha}$ and the first relation in (2) yields the requested equality (3).

3.8. Theorem. *For any ordered pair* (a,b) *of cardinal numbers* a, b *there exists a family* F *of sets and some ordinal number* α *such that* $a = cF$ *and* $cF^{I\alpha} \geqslant b$.

As a matter of fact, we can consider any disjoint family F *of cardinality* a; *then for some* α *we have* $a^{k\alpha} \geqslant b$ *and consequently* $(cF)^{k\alpha} \geqslant b$.

3.9. Theorem. *For any* F *and any sets* A, B *we have*

$$kA = kB \Rightarrow cF^{\cdot A} = c(F^{\cdot B}) \text{ and}$$

$$kA \leqslant kB \Rightarrow cF^{\cdot A} \leqslant c(F^{\cdot B}).$$

As a matter of fact let t be a one-to-one mapping of A into B; and let D be a disjoint system in $F^{\cdot A}$; for $f \in D$ we define $t = t(f)$ in this way

$$f: A^{\cdot} \to fA^{\cdot}, \text{ where } f\ A^{\cdot} \in F;$$

the antidomain tA of t is a part of B; to every mapping $f: A \to F$ we define the mapping $v_f: B \to F$ as the one which equals ft^{-1} in tA and which, in $B \setminus tA$, equals a constant $b \in B \setminus tA$. Then $v_f \in F^{\cdot B}$.

To every disjoint set D in $F^{\cdot A}$ corresponds an equivalent system $v_D = \{v_f : f \in D\}$ in $F^{\cdot B}$. If $tA = B$, then the mapping $f \to v_f$ is an isomorphism from $F^{\cdot A}$ onto $F^{\cdot B}$.

3.10. Theorem. *Let* A, B *be any sets and* F *a family of sets; then* $cF^{\cdot(A \cup B)} \leqslant s^i$, *where* $s = \sup \{a, b\}$, $i = \inf \{a, b\}$, $a = cF^{\cdot A}$, $b = cF^{\cdot B}$. *If the product* ab *is infinite, then* $cF^{(A \cup B)} \leqslant 2^{\sup \{a, b\}}$.

We shall consider the case that A, B are non empty disjoint sets. Then every member $x \in F^{\cdot(A \cup B)}$ is the set of the functions $g \mid (A \cup B)$ where for $i \in A \cup B$ one has $g_i \in x_i$, x_i being a member of F. Let g_A be the corresponding subfunction in A and let x_A be the set of all these subfunctions g_A; analogously one has g_B and x_B. For any set $S \subseteq F^{\cdot(A \cup B)}$ one has the „projections"

$$S_A = \{x_A;\ x \in F^{\cdot(A \cup B)}\},\ S_B = \{x_B;\ x \in F^{\cdot(A \cup B)}\}.$$

In particular, for every disjoint set D in $F^{\cdot(A \cup B)}$ we have D_A, D_B. For any members X, Y of $F^{\cdot(A \cup B)}$ the relation $X \cap Y = \varnothing$ means

(1) $$X_A \cap Y_A = \varnothing \text{ or } x_B \cap y_B = \varnothing. \quad (2)$$

Let $X \rho Y$ mean (1) i. e. that $X_A \cap Y_A = \varnothing$. Then D is a graph relative to the relation ρ. Let L be a ρ-chain in D; this means that

(3) $$\{X, Y\}_{\neq} \subseteq L \Rightarrow \{X_A, Y_A\} \subseteq L_A \text{ and } X_A \cap Y_A = \varnothing$$

and that L_A is a disjoint system in D_A. Now, the family D_A is isomorphic to a subsystem of the family $F^{\cdot A}$, therefore

$$(4) \qquad k\,L_A \leqslant a\,(=c\,F^{\cdot A}).$$

Because of the relations (3) the correspondence $X \in L \rightarrow X_A \in L_A$ is onto and one-to-one:

$kL = kL_A$ what jointly with (4) yields

$kL \leqslant c\,A$ for every ρ-chain L in the graph $(A;\ \rho)$.

Analogously, one proves that every antichain M of $(D,\ \rho)$ yields disjointed system M_B of cardinality kM; since $k\,M_B \leqslant c\,F^{\cdot B}$ this means that every antichain in $(D,\ \rho)$ is of a cardinality $\leqslant b = c\,F^{\cdot B}$. Consequently, by the graph -chain-antichain-theorem we have the requested relation.

3.11. Theorem. *If r is a natural number and F a set family then*

$$(1) \qquad c\,F^{\cdot Ir} = 2^{cF} \Rightarrow c\,F^{\cdot I\,(r+n)} = c\,F^{\cdot Ir} \qquad (2)$$

for every integer n.

Proof. The proof is carried out by induction relative to n. Let D be a disjointed system in $F^{\cdot I\,(r+1)}$; then

$$\{X,\ Y\}_{\neq} \subseteq D \Leftrightarrow$$

$$(3) \qquad (X_0 \times X_1 \times \ldots \times X_{r-1}) \cap (Y_0 \times Y_1 \times \ldots \times Y_{r-1}) = \varnothing\ \vee$$

$$\vee\ X_r \cap Y_r = \varnothing. \qquad (4)$$

Let $X\,\rho\,Y$ mean that (3) occurs; then to every ρ-chain $L \subseteq D$ corresponds the disjointed chain L_{Ir}-projection of L into the product $F^{\cdot Ir}$; consequently $k\,L_{Ir} \leqslant c\,F^{\cdot Ir}$ and according to the assumption (1) we have $k\,L_{Ir} \leqslant 2^{cF}$; again $k\,L = k\,L_{Ir}$ and thus $k\,L \leqslant 2^{cF}$. Consequently, every chain of the graph $(D,\ \rho)$ is $\leqslant 2^{cF}$. Analogously, one proves that every antichain of (D, ρ) is $\leqslant cF$. Hence $k\,D \leqslant (2^{cF})^{cF} = 2^{cF}$.

The implication (1)$\Rightarrow$(2) is thus proved for every r and $n = 1$; writing in particular $r+1, r+2, \ldots$ instead of r, the implication (1)$\Rightarrow$(2) is proved for $n = 1, 2, 3, \ldots$ i. e. for every n.

3.12. Problem. *Let F be a system of sets and n a natural number satisfying $\aleph_0 \leqslant c\,F^{\cdot In} = c\,F^{\cdot (n+1)}$; is there one-to one mapping of $F^{\cdot I(n+1)}$ into $F^{\cdot In}$ which conserves both disjointness and jointness of sets? In other words, is then the disjonction graph $(F^{\cdot I(n+1)}; D)$ isomorph to a subgraph of $(F^{\cdot I(n)}; D)$?*

§ 4. Disjoint systems in $F_1 \cdot \times \cdot F_2$.

4.1. F_1, F_2 being set families let Δ be a disjoint system (or D-chain) in the product

$$(1) \qquad F_1 \cdot \times \cdot F_2 = \{x_1 \times x_2;\ x_1 \in F_1 \wedge x_2 \in F_2\}.$$

Let $pr_1\Delta = p_1\Delta$ and $p_2\Delta$ be the first and the second projection of Δ respectively. For any $a_1 \in pr_1\Delta$ we have the following antiprojection of a_1 into Δ:

$p_1^{-1}\Delta(a,.)=\{(a,y);\ y\in F_2\ (a,y)\in\Delta\}$; for any subset $A\subseteq F_1$ we have the corresponding first antiprojection of A into Δ defined by

$$p_1^{-1}\Delta(A,.)=\bigcup_{a\in A}\{p_1^{-1}\Delta(a,.)\}.$$

Analogously, one defines the second antiprojection of any $B\subseteq F_2$ in this way:

$$p_2^{-1}\Delta(.,B)=\bigcup_{b\in B}\{p_2^{-1}\Delta(.,b)\},\ \text{where}$$

$$\{p_2^{-1}\Delta(.,b)\}=\{(x,b);\ x\in F_1,(x,b)\in\Delta\}.$$

By an argument we used in section 3 one proves readily the following items.

4.2. Lemma. For every $a_1\in F_1$ the first antiprojection $p_1^{-1}\Delta(a_1,.)$ in Δ yields the disjoint second projection $p_2 p_1^{-1}\Delta(a_1,.)$; therefore the cardinality of this set as well as that of $p_1^{-1}a_1$ is $\leqslant cF_2$. The first antiprojection in Δ of any jointed system C in F_1 is a disjoint system in F_1; the p_2 — projection of $p_1^{-1}C$ is a one-to-one mapping yielding a disjoint system of cardinality $\leqslant cF_2$ in F_2.

4.3. Lemma. Let $T=T(\Delta_1)$ be any tree or ramified table of the family $(F,\supseteq)$; then every D-chain in T is $\leqslant cF_1$ and every J-chain of T is $\leqslant cF_2$. If the number $s=\sup\{cF_1,\ cF_2\}$ is infinite, then one knows that

(1) $kT\leqslant s^{\varepsilon}$; where $s^{\varepsilon}\in\{s,\ s^+\}$; in particular the tree hypothesis yields $s^{\varepsilon}=s=cF_1\cdot cF_2$ and therefore

(2) $kT\leqslant cF_1\cdot cF_2$.

4.4. For every $x_1\in\Delta_1$ let

(3) $\Delta_1(.,\ x_1]_{\geqslant q}$; denote the system of all the members of Δ_1, each joint with x_1 and none contained as a proper part of x_1; then one has the star number $S\Delta_1(.,\ x_1]$ as the minimal number of chains in (3) exhausting (3). Each J-chain in (3) being $\leqslant cF_2$ (Lemma 4.2), we infer that

(3) $k\Delta_1(.,\ x_1)_q\leqslant cF_2\cdot S\Delta_1(..\ x_1)_q,\ (x_1\in\Delta_1)$ and hence

(4) $k\Delta_1(.,\ x_1]_q\leqslant cF_2\cdot s_1F_1$, where

(5) $s_1F_1=\sup\limits_{x\in F_1}S(.,\ x]_q$; the number s_1F_1 is called the *left local star number of the family* $(F_1;\supseteq)$.

4.5. Now as consequence of the choice axiom it is easy to prove the existence of a subtree $T=T(\Delta_1)$ in Δ_1 that is *quasi-cofinal* with Δ_1 in the sense that[1]

(6) $$\Delta_1=\bigcup_{t\in T}\Delta_1(.,\ t)_{\geqslant q};$$

this means that to every $x\in\Delta_1$ corresponds some $t\in T$ such that x meets t but is not a proper part of t^1. By induction the rows $T_0, T_1,\ldots$ of such a T are defined in this way; T_0 is any maximal disjoined system in Δ_1; for every $t_0\in T_0$ let ft_0 be any *maximal* disjoint system in Δ_1, each member of ft_0 being a proper subset of t_0; one puts

$$T_1=\cup ft_0,\ (t_0\in T_0)\ \text{etc.}$$

Putting for every ordinal α

$$T^{\alpha}=\cup_{\xi}T_{\xi}\,(\xi<\alpha)$$

[1] This fact was found also by S. Mardešić.

one sees that T^α is a tree; if T is quasi-cofinal with Δ_1, we put $T^\alpha = T$; if T^α is not quasi-cofinal to Δ_1, we construct T_α as $\cup f t_{\alpha-1} (t_{\alpha-1} \in T_{\alpha-1})$ if the ordinal α is of the first kind; if α is a limit ordinal >0, we consider every jointed decreasing α— sequence C^α of members of T^α and consider any maximal disjoint system $f C^\alpha$ of $\Delta_1 \setminus \cup \Delta_1 (. , t)_{\geqslant q}$, each being a proper part of every member of C^α. One puts then

$$T = \bigcup_{C^\alpha} f C^\alpha$$

(one sees that the construction for non limit α is reducible to this construction using C^α, because for every $t_{\alpha-1} \in T_{\alpha-1}$ the α—sequence of oversets of $t_{\alpha-1}$ in T^α is such that $f t_{\alpha-1}$ serves as $f C^\alpha$).

4.6. This being done let $T(\Delta_1)$ be any quasi-cofinal subtree of Δ_1. The decomposition (6) yields jointly with (4)

$$k \Delta_1 \leqslant k T \cdot c F_2 \cdot s_1 F_1. \tag{7}$$

The relation (7) by (1) yields

$$k \Delta_1 \leqslant c^\varepsilon \cdot c F_2 \cdot s_1 F_1. \tag{8}$$

Going back from Δ_1 to Δ the relation (8) in virtue of Lemma 4.2. gives

$$k \Delta \leqslant c F_1 \cdot s^\varepsilon \cdot c F_2 \cdot s_1 F_1. \tag{9}$$

This holding for every D—chain Δ of $F = F_1 \cdot \times \cdot F_2$ one concludes that

$$c F (= \sup_{\Delta \subset F} k \Delta) \leqslant c F_1 \cdot s^\varepsilon \cdot c F_2 \cdot s_1 F_1. \tag{10$_1$}$$

4.7. Analogously, considering the second projection Δ_2 of Δ one proves that

$$c (F_1 \cdot \times \cdot F_2) \leqslant c F_2 \cdot s^\varepsilon \cdot c F_1 \cdot s_1 F_2. \tag{10$_2$}$$

4.8. The relations (10_1), (10_2)yield by multiplication:

$$(c F)^2 \leqslant (c F_1 \cdot c F_2)^2 \cdot (s^\varepsilon)^2 \cdot z_1 \ F_1 \cdot s_1 F_2. \tag{11}$$

4.9. If s is infinite, then cF is infinite also and $(cF)^2 = cF$ and the exponents 2 in (11) could be dropped; we obtain

$$c F \leqslant s \cdot s^\varepsilon \cdot s_1, \text{ where } s_1 = \sup \{s_1 F_1, \ s_1 F_2\}.$$

Since $s \leqslant s^\varepsilon$ we have $s \cdot s = s^\varepsilon$ and consequently

$$c \leqslant s \cdot s_1.$$

Since obviously $c_1, \ c_2 \leqslant c$ thus $s \leqslant c$ and we have proved the following relation

$$s \leqslant c F \leqslant s \ \cdot s_1. \tag{12}$$

4.10. Theorem. (I) *Let I be a finite index set (e.g. the interval In of ordinals $< n$, where n is a given finite ordinal); let F_i, $(i \in I)$ be a finite sequence of non void set systems such that at least one of the cellular numbers cF_i be infinite; then*

$$s \leqslant c \prod_i F_i \leqslant s^\varepsilon \cdot s_1, \textit{ where } s = \sup_i \ c F, \ s_1 = \sup_i \ s_1 F_i, \textit{ and } s^\varepsilon \in \{s, s^+\}.$$

In particular for any set system G and any natural number n one has

$$c\,G \leqslant c\,(G^n) \leqslant (c\,G)^{\varepsilon} \cdot s_1.$$

(II) One has $s^{\varepsilon} = s$ if and only if the tree hypothesis

$$k_c\,T \leqslant s \text{ and } k_c\,T \leqslant s \Rightarrow k\,T \leqslant k_c\,T \cdot k_{c'}\,T$$

is true or false.

Since the theorem (II) was proved else (G. Kurepa, [1] p. 106 theor. 1), let us prove the theorem (1).

We just proved that the theorem (I) holds if the index set I has 2 members; by induction argument one sees that the same conclusion holds for any finite set I.

Let us prove the theorem (I) if I has 3 membres 1,2,3. Let Δ be any D-chain in $F (= F_1 \times F_2 \times F_3)$ and Δ_{12} and Δ_3 its projections into $F_1 \cdot \times \cdot F_2$ and F_3 respectively. For any tree T in Δ_{12} that is quasi—cofinal with Δ_{12} we have (like in (7)):

$$k\,\Delta_{12} \leqslant k\,T \cdot c\,F_3 \cdot s_1\,(F_1 \cdot \times \cdot F_2). \tag{13}$$

We have to evaluate the factors $k\,T$, s_1 in (13). First of all,

$$s_1\,(F_1 \cdot \times \cdot F_2) \leqslant s_1\,F_1 \cdot s_1\,F_2. \tag{14}$$

As a matter of fact, for any $x_1 \in F_1$ let A_1 be a system of J-chains of sets $\in F_1$ exhausting $F_1\,(.\,,x_1)_{\supseteq q}$; analogously, for $x_2 \in F_2$ one has a family A_2 of jointed systems of sets-members in $F_2\,(.\,,a_2)_{\supseteq q}$ exhausting this family. Then we have the element $x_1 \times x_2 \in F_1 \cdot \times \cdot F_2$ and the system $A_1 \cdot \times \cdot A_2$; all elements of this system are J-chains, each quasi-containing $x_1 \times x_2$ and the system exhausts $F_0 = F\,(.\,,x_1 \times x_2)_{\supseteq q}$; because if $M_1 \times M_2$ is any member of F_0 q-containing $x_1 \times x_2$, then M_i q-contains x_i and for some $J_i \in A_i$ we have $M_i \in J_i$ and hence $M_1 \times M_2 \in J_1 \times J_2 \in A_1 \cdot \times \cdot A_2$. In this way we proved that

$$s\,(F_1 \times F_2)\,(.\,,x_1 \times x_2)_{\supseteq q} \leqslant s\,F_1\,(.\,,x_1)_{\supseteq q} \cdot s\,F_2\,(.\,,x_2)_{\supseteq q};$$

from here allowing x_1, x_2 to vary in F_1, F_2 respectively and taking sup we have the requested relation (14).

The relations (13), (14) yield

$$k\,\Delta_{12} \leqslant kT \cdot cF_3 \cdot s_1\,F_1 \cdot s_1\,F_2. \tag{15}$$

4.11. Lemma. Let $s_{12} = \sup\{cF_1, cF_2\}$; every chain and every antichain of every tree $T \subset F_1 \cdot \times \cdot F_2$ is $\leqslant s_{12}^{\varepsilon}$; also $kT \leqslant s_{12}^{\varepsilon}$.

In opposite case there would be a tree T_a in F of cardinality $\geqslant s_{12}^{\varepsilon+}$; now obviously, $w_d\,(F_1 \cdot \times \cdot F_2) \leqslant w_d\,F_1 \cdot w_d\,F_2$, where $w_d\,A$ for any family A of sets denotes the supremum of cardinalities of strictly decreasing sequences of sets in A. Since kT_a is greater than the cardinality of any tree in F_1 or in F_2, and since this fact is not due to J-subchains of trees, it should be due to D-subchains, and the tree should contain an antichain $\geqslant {}_{12}^{\varepsilon+}$, in contradiction with 4.3.

The relation (15) and the Lemma 4.11. imply

$$k\,\Delta_{12} \leqslant s_{12}^{\varepsilon} \cdot cF_3 \cdot s_1\,F_1 \cdot s_1\,F_2.$$

From here, going back to Δ:

$$k\Delta \leqslant \left(s_{12}^{\varepsilon}\, cF_3 \cdot s_1 F_1 \cdot s_1 F_2\right) \cdot cF_3. \tag{16}$$

Now, let $s = \sup_i cF_i$ and $s_1 = \sup_i s_1 F_1$; then $s_{12} \leqslant s$, $cF_3 \leqslant s$ and therefore $s_{12}^{\varepsilon}(cF_3)^2 = s^{\varepsilon}$; again $s_1 F_1 \cdot s_1 F_2 \leqslant s_1$ and the relation (16) yields

$$k\Delta \leqslant s^{\varepsilon} s_1.$$

This proves the theorem for $I = 1, 2, 3$. By induction argument one proves the theorem for every finite index set I. *Q. E. D.*

The foregoing theorem, by particularization implyes the following.

4.12. Theorem. *Let I be a finite index set and $F_i (i \in I)$ a sequence of non void set systems with sup $cF_i = s = \infty$; if $s_1 F_i \leqslant s$, then*

$$s \leqslant c \prod_i F_i \leqslant s^{\varepsilon}.$$

Such a case holds particularly if F_i is a system of intervals of a totally ordered set $O_i (i \in I)$; in this case one has $s_1 F_i \leqslant 2$.

As a matter of fact any system S of intervals overlapping a given interval x of a given totally ordered set equals $S_1 \cup S_2$, where S_1 denotes all the members of S containing the left extremity of x and where S_2 denotes all the members of S each containing the right extremity of x; obviously, S_i is jointed.

§ 5. Cartesian multiplication of topological spaces

5.1. Definition of cartesian multiplication of spaces[1]. For every $i \in I$ let X_i be a topological space; the cartesian product X of sets X_i of points of X_i will be called the *topological product of spaces* X_i provided for every point $x \in X$ the neighbourhoods are defined in the following way; let I_0 be a *finite part of I;* for every $i_0 \in I_0$ let $O(i_0)$ be a neighbourhood of the point x_{i_0} in the space X_0; for every $i \in I$ let X_i^{+} be $O(i_i)$ or X_i, according as $i \in I_0$ or $i \in I \setminus I_0$; the cartesian product of all the sets X_i^{+} is called neighbourhood of the point x. This neighbourhood depends on finite set $I_0 \subseteq I$ and on the neigbourhoods $O(i_0)$ in X_{i_0} for $i_0 \in I_0$. The stress in the foregoing definition is the finiteness of subsets I_0 of I.

5.2. The neighbourhoods could be defined in this way also. For a point x_i on the i^{th} coordinate axis let $p_i^{-1}(x_i)$ be the i^{th} antiprojection of x_i into the space i. e. the set of all the points x of the space, the i^{th} coordinate of which is just the point x_i of the space X_i. For a subset S_i of the space X_i we define the i^{th} antiprojection $p_i^{-1} S_i$ as the union of all the sets $p_i^{-1} x_i (x_i \in S_i)$. In other words, the set $p_i^{-1} S_i$ is the anti-projection of S_i in the direction of the X_i-axis. Then the foregoing neighbourhood is the intersection of the open sets like this

$$\cap\, p_{i_0}^{-1}(0 x_{i_0})\ (i_0 \in I_0). \tag{1}$$

[1] We shall consider topological T_2-spaces i. e. Fréchet's V-spaces satisfying the Hausdorff's T_2-condition of separation. The T_2-separation condition means that for any 2-point set $\{a, b\}$ there is a neighbourhood $V(a)$ of a and a neighbourhood $V(b)$ of b such that $V(a) \cap V(b) = \varnothing$.

Marczewski-Szpilrajn [1] proved that if the topological spaces X_i are of a countable weight each, then the weight of the cartesian product X is countable too. The phenomenon is a general one and we have the following.

5.3. Theorem. *For every topological T_2-spase S and every non void index set I the cellularity of the cartesian hyper-cube S^I equals $(cS)^{kI}$ for w. $kI < \aleph_0$; if $kI \cdot w \geqslant \aleph_0$, then $\sup\{\aleph_0, cS\} \leqslant cS^I \leqslant w$, where the weight w $(=wS)$ of the space S is defined as the infimum of cardinal numbers of neighbourhood bases of the space S.*

The theorem 5.3. is a special case of the following theorem (in the wording of the theorem put X_i = fixed space S for every $i \in I$).

5.4. Theorem. *Let I be a non void set and X_i, for every $i \in I$, a topological space. Let wX_i denote the weight number of the space X_i and $w = \sup wX_i$; then for the cellularity number cX of the cartesian product $X = \prod_i X_i$ one has:*

(2)
$$kI \cdot w < \aleph_0 \Rightarrow \prod_i cX_i = cX$$

(2′)
$$kI \cdot w \geqslant \aleph_0 \Rightarrow \sup\{\aleph_0, \sup_i cX_i\} \leqslant cX \leqslant w.$$

5.5. Proof of the theorem 5.4.

1. First case: The number of factors X_i is finite and every X_i is finite. In this case, obviously $cX_i = kX_i = wX_i$ and $cX = kX = wX$; since $kX = \prod_i kX_i$, the preceding relations yield the requested implication (2).

Second case: $kI \cdot w \geqslant \aleph_0$. Now, it is obvious that if kI is infinite and every factor has at least 2 points, then the number c $(=cX)$ can not be finite. Therefore we have still to consider the case that the weight of every factor is infinite, irrespective what happens with kI.

Let c denote the number cX. First of all, $c \geqslant cX_i$ for every $i \in I$ and hence $c \geqslant c_s$ $(=\sup cX_i)$. As a matter of fact, let D_i be any disjoint system of open sets of the space X_i; putting $\{i\} = I_0$ and taking $O_i \in D_i$, $O_j = X_j$ for $j \in I \setminus \{i\}$, one gets a system of cardinality kD_i of open sets $\prod_{x \in I} O_x$ of the space X; therefore $c \geqslant kD_i$ and $c \geqslant cX_i$ $(= \sup_{Di \subset Xi} kD_i)$ for every $i \in I$. The relations $c \geqslant cX_i$ imply $c \geqslant \sup c_i$ i. e. $c \geqslant c_s$. Therefore the requested relation (2′) will result of the impossibility of the relation $c > w$. Obviously we can suppose also that $w \geqslant \aleph_0$.

5.5.2. Now, suppose on the contrary that $c > w$ and that there exists a *disjoint* system D of open sets of the space X such that

(2)
$$kD > w \geqslant \aleph_0.$$

One might suppose that the members of D are of the form (1), where I_0 is a (variable) finite subset of I (it is sufficient to choose an element of the form (1) in each member of D and consider the system of the selected elements). For every $i \in I$ let B_i be a basis of neighbourhoods of the space X_i such that

(3)
$$kB_i = wX_i \ (:= w_i).$$

This being done, let n be any natural number and D_n the system of all the members (1) in D such that $kI_0 = n$. Obviously $D = \cup_n D_n$ $(n < \omega)$ and since, by (2), the system D is non countable there exists an integer m such that also

(4)
$$kD_m > w.$$

5.5.3. Let $x = \prod X_i^+$ $(i \in I)$ be a particular member of D_m; then the set I_{om} of the points $i \in I$ such that $X_i^+ \neq X_i$ is a well determined *finite* subset of m points of I. The members of D_m being pairwise disjoint one has in particular $x \cap y = \varnothing$ for every $y \in D_m \setminus \{x\}$; this means that for every such y one has $x_i \cap y_i = \varnothing$ for some $i = i(y) \in I_{om}$ because $x_i \cap y_i \neq \varnothing$ for every $i \in I \setminus I_{om}$. This mapping

$$(5) \qquad f: D_m \to I_{om}$$

is a single-valued mapping of the set D_m of cardinality $> \aleph_o$ into a *finite* m-point set $I_{om} = \{i_1, i_2, \ldots, i_m\}$. Therefore for some $j_1 \in I_{om}$ there exists a subset D_m^o of D_m in which the mapping (4) equals j_1 and so that $k D_m^o = k D_m$. Now, let us consider the p_{j_1} — projection of the set D_m^o into the space X_{j_1}; this mapping is a single-valued mapping of the set D_m^o of cardinality $> w$ into the basis B_{j1} of w_{j1} members of the space X_{j1}. Since $k D_m^o > w \geqslant w_{j_1}$, one infers that for some member $O_{j'_1} \in B_{j_1}$ and for some subset D_{m_1} of D_m^o one would have

$$pr_{j_1} D_{m_1} = O_{j'_1}, \quad k D_{m_1} > w.$$

5.5.4. Substituing D_{m_1} for D_m and $I_{om} \setminus \{j_1\}$ for I_{om} the argument of 4.4.3. shows that for: some point $j_2 \in I_{om} \setminus \{j_1\}$, some subset D_{m2} of D_{m1} and some neighbourhood $O_{j'_2} \in B_{j_2}$ one has:

$$f | D_{m2} = j_2, \quad pr_{j_2} D_{m2} = O_{j'_2}, \quad k D_{m2} > w.$$

The induction procedure would go on: there woud be a subset D_{m3} of D_{m2}, a point $j_3 \in I_{om} \setminus \{j_1, j_2\}$ and a $O_{j'_3} \in B_{j_3}$ such that

$$f | D_{m3} = j_3, \quad pr_{j_3} D_{m3} = O_{j'_3}, \quad k D_{m3} > w; \text{ etc.}$$

The m^{th} step of induction procedure would yield: a disjoint subset D_{mm} of $D_{m\,m-1}$, a point $j_m \in I_{om} \setminus \{j_1, j_2 \ldots, j_{m-1}\}$ and a member $O_{j'_m} \in B_{j_m}$ such that

$$(6) \qquad f \setminus D_{mm} = j_m, \quad pr_{j_m} D_{mm} = O_{j'_m}, \quad k D_{mm} > w.$$

Now, for every index $i \in I \setminus I_{om}$ we have $pr_1 D_{mm} = X_i$; therefore $x \in D_{mm} \Rightarrow X_{j\mu}^+ = O_{j'\mu}$ for $\mu = 1, 2, \ldots, m$ and $X_i^+ = X_i$ for $i \in I \setminus I_{om}$; consequently $k D_{mm} = 1$, in contradiction with the last relation in (6). This contradiction proves the theorem.

5.5.5. Remark. Let us consider the cellularity numbers $cF^{\cdot I\alpha}$ (F and α being any system of sets, and any ordinal number) and the numbers $cS^{I\alpha}$ (S being any topological space); in the first case, as α is increasing so is also the corresponding cellularity; on the contrary, in the case of hyper-cubes of any topological space S the cellularity numbers are always less or equal to the weight wS of S.

5.5.6. Corollary. *For any topological T_2-space S satisfying $cS = wS$ one has $cS^I = cS$ for every non void set I. In particular, $cS^\alpha = cS$ for any positive ordinal α, irrespective whether α is finite or transfinite. In particular this holds provided S is a metrical space or if S is a totally ordered space in which the cellularity equals the separability number of the space.*

A consequence of the corollary 5.5.6. for metrical spaces is this one:

5.5.7. *The hypercube M^I of any metrical space M with $kI > \aleph_0$ is a non metrical space; in particular the real cube $[0, 1]^{I(\omega_1)}$ or $\{0, 1\}^{I(\omega_1)}$ are non metrical spaces.*

5.5.8. Corollary. *For every topological space S the set consisting of the cellularity numbers $c\,S^{\alpha}$ running through ordinal numbers is well determined and has at most wS numbers (cf. theorems* 3.7; 3.8; 5.8 *(ii)).*

5.6. *Comparison between cS* and *cS^n for any space S.*

Theorem. (1). *For any topological space S one has* $cS \leqslant cS^2 \leqslant \inf\{2^{cS}, (cS)^{\varepsilon}\cdot sS\}$; *for every ordinal $n<\omega$.*

(II) *For any ordered pair of topological spaces S_1, S_2 we have* $s \leqslant c(S_1\times S_2) \leqslant 2^s$, $Tr(S_1)\, s_1 S_1$, $Tr(S_2)\cdot s_1 S_2$ *where* $s=\sup\{cS_1, \; cS_2\}$: $Tr\, S_1 = \inf kT$, T being quasi-cofinal subset of the family of open sets of S_1 (cf. § 4.4 and § 4.5).

(III) *For any ordered pair of totally ordered spaces S_1, S_2 one has* $s \leqslant c(S_1\times S_2) \leqslant s^{\varepsilon}$, *where* $s^{\varepsilon}\in\{s, s^+\}$; *the relation $s^{\varepsilon}=s$ is equivalent to the tree hypothesis.*

The proof is like the one of theorem 3.4. (1) in § 3.5.2; cf also § 3.10 and § 4.9; § 4.10.

5.7 In connexion with the results 3.4. (1), 4.4 and 4.6 let us indicate that there are spaces $S_{\aleph_1}$ satisfying $cS<wS$; such a space is the cartesian product $[0,1]^{\aleph_1}$ of $\aleph_1$ real segments [0,1]; the cellularity and the weight of this product are $\aleph_0$, $\aleph_1$ respectivaly.

5.7.1. Theorem. For any ordered pair (a,b) of cardinal infinite numbers there is a topological space S such that $cS \leqslant a < b \leqslant wS$. Such a space is the cartesian product of $k\alpha$ real segments [0,1] where α is any ordinal of cardinality $\geqslant b$.

5.7.2. Here is also a space S satisfying $cS < wS$ and which was given by Inagaki as the solution of a problem in my doctoral thesis. Let R^* be the set of members of a one-to-one ω_1— sequence x_α $(\alpha<\omega_1)$ of real numbers x_α; the set R^* is topologized by considering for any $\alpha<\omega_1$ as neighbourhoods of x_α the sets of the form $V^*(x_\alpha)=\{x_\beta;\; x_\beta\in V(x_\alpha);\; \alpha\leqslant\beta<\omega_1\}$, $V(x_\alpha)$ being any ordinary neighbourhood of x_α. The space (R^*, V^*) so obtained has the cellularity $\aleph_0$ and the weight $\aleph_1$.

5.8. Main theorem (I). *For any topological space S satisfying $cS \geqslant \aleph_0$ and every index set I the cellularity of the cube S^I is $\leqslant 2^{cS}$ i. e.*

$$\text{cel}\, S \leqslant \text{cel}\, S^I \leqslant (\text{cel}\, S)^{\text{cel}\, S}$$

(ii) *The general continum hypothesis implies*

$$\text{cel}\, S^I \in \{\text{cel}\, S, \; (\text{cel}\, S)^+\}.$$

5.8.1. Proof. First of all we proved that the theorem (1) holds provided $kI=2$, e. g. $I=\{1, 2\}$ and even for $kI<\infty$ (cf theorem 3.11).

Now we shall prove the theorem (1) for every I.

5.8.2. Lemma. *Let m be any positive integer $1<m\leqslant kI$ and Δ any disjoint system of open sets of S^I satisfying $ksx=m$ for every $x\in\Delta$; then $k\Delta \leqslant 2^{cS}$; here sx denotes the greatest subset I_0 of the index set I satisfying $x(i)\neq S\;(i\in I_0)$.*

We shall prove the lemma by induction argument on m. Suppose that the lemma holds for every natural number $<m$; let us prove that it holds also for m. Assume on the contrary that there exists a disjoint system Δ of cardinality $>2^{cS}$ and such that $ksx=$ m for every $x\in\Delta$. Since the set Δ is *disjoint*, the set of $sx\,(x \in \Delta)$ is *jointed*; namely, if $x,\ y \in \Delta,\ x \neq y$, then $x \cap y = \varnothing$ what means that for some $i \in I$ one has $x_i \cap y_i = \varnothing$; this means that

$$x_i \neq S \neq y_i \text{ i. e. } i \in sx \cap sy, \text{ hence } sx \cap sy \neq \varnothing.$$

Now, let $e \in \Delta$; for every $a \in se$ let $\Delta(a)$ be the system of all the members x of Δ satisfying $a \in sx$. Then

$$\Delta = \bigcup_a \Delta(a) \quad (a \in se).$$

Since the set se has just m members, one of the sets $\Delta(a)$ has $k\Delta$ points; let a_0 be such an element of se:

$$k\Delta(a_0) = k\Delta > 2^{cS} \quad \text{and} \quad a_0 \in se.$$

Let us consider the *disjoint* set system $\Delta(a_0) = A$. Let us structurize A by defining that for $x,\ y \in A$ the relation xry means $x_i \cap y_i = \varnothing$ for some $i \in sx \cap sy \setminus \{a_0\}$.
Let L be a r-chain in $(A;\ r)$; then L is a disjoint set in S^I; moreover, let L_I be the system of sets x_0 where for every $x \in L$ one denotes by x_0 the set obtained from x by substituing the a_0-factor of x by the space S. The mapping $x \in L \rightarrow x_0$ is one-to-one; let $L_0 = \{x_0;\ x \in L\}$. Then L_0 is disjointed system of sets of S^I such that

(1) $$k\,s\,x_0 = m - 1 \quad (x_0 \in L_0). \text{ Namely, } s\,x_0 = s\,x \setminus \{a_0\}.$$

Now, by induction hypothesis the relations (1) imply that $kL_0 \leqslant 2^{cS}$, what jointly with $kL_0 = kL$ implies the requested relation $kL \leqslant 2^{cS}$. In other words every r-chain L in $(A,\ r)$ is $\leqslant 2^{cS}$; therefore also the k_c-number of $(A,\ r)$ is $\leqslant 2^c$. On the other hand every antichain L' in $(A,\ r)$ is $\leqslant cS$ because if $x,\ y \in L'$ and $x \neq y$, then

$$x_i \cap y_i \neq \varnothing \text{ for } i \in sx \cap sy \setminus \{a_0\} \text{ and therefore } x_{a_0} \cap y_{a_0} = \varnothing.$$

In other words to every antichain L' in $(A,\ r)$ corresponds a well determined disjoint system in S, of cardinality kL'.
In virtue of the chain-antichain theorem for graphs one concludes that $k\Delta \leqslant (2^{cS})^{cS} = 2^{cS}$ i. e. $k\Delta \leqslant 2^{cS}$. *Q. E. D.*

5.8.3. Proof of the theorem (I). For any natural number n let

$$\Delta_n = \{x;\ x \in \Delta,\ k\,s\,x = n\}.$$

Then the sets Δ_n exhaust Δ; since by hypothesis $k\Delta > 2^{cS}$, then for some integer n one would have necessarily $k\Delta_m > 2^{cS}$, in contradiction with the foregoing lemma, because every member x of Δ_m satisfies $k\,s\,x = m$.

5.8.4. The theorem 5.8. (ii) is an obvious consequence of the theorem 5.8. (i) and of the general continuum hypothesis.

6. On the cartesian multiplication of ordered sets and graphs

6.1. Definition. Let (1) $(O_i, <_i)$ $(i \in I)$ be a family of ordered sets; the *cartesian product* or the *cardinal product* of the sets (1) is the set $(O, <)$ where $O = \prod_i O_i$ $(i \in I)$ and where for $x,\ y \in O$ one has

$$x \leqslant y \text{ in } O \Leftrightarrow x \leqslant_i y \text{ in } O_i \ (i \in I).$$

Analogously one defines the cartesian product of any non empty family of graphs (G_i, r_i) on substituing in the preceding definition G_i for O_i and r_i for $<_i$; r_i means any binary relation that is either symetrical (for *symetrical graphs*) or antisymetrical (for *oriented graphs*).

One proves readily the following.

6.2. Theorem. The cartesian product of any system of ordered sets is an ordered set; if every factor is ramified (a tree, a chain), the product need not be so.

We are especially interested to know the connexions between the cellularity number of the product and the cellularity numbers of the factors. In this respect the notion of ramified sets and particularly of ramified tables or trees is of special importance.

6.3. Definition of a node. Every *maximal* subset S of a ramified set R such that

(1) $$x, y \in S \Rightarrow S(. , x) = S(. , y)$$

is called a *node of S*.

6.4. Theorem. *Let R be any ramified set i.e. any ordered set $(R; <)$ in which $R(. , R^{\cdot})$ is a chain; let I be any non empty index set; let $f, g \in R^I$; then $f \leqslant g$ in R^I means $f_i \leqslant g_i$ in R for every $i \in I$; if $f \neq g$ and if the set $fI = \{f_i;\ i \in I\}$ lays in a node of R as well as does gI and if*

$$kfI > 1, \quad kgI > 1$$

then $f \parallel g$ i. e.: neither $f \leqslant g$ nor $f \geqslant g$. The conclusion holds also provided R contains a subset M such that fI lays in a node of M and that gI lays in a node of M.

Proof. Since by hypothesis the set fI is located in a node of R, the chain $R_f = R(. , f_i)$ is well determined and does not depend on an particular choice of i in I. Analogously, one has $R_g = R(. , g_i)$ for $i \in I$. Let

(2) $$C = R_f \cap R_g.$$

The set C is a chain in R and is an initial section of the chains R_f, R_g.

Case (i). C is a proper subset of both R_f and R_g. Then there is an $a \in R$ and $a, b \in R$ such that

$$a \| b, C. < a < f_i \text{ and } C. < b < g_i.$$

Hence $f_i \| g_i$, for one has not e. g. $f_i < g_i$, because the antichain $\{a, b\}$ would be $< g_i$, contrary to the ramification condition on R. Thus $f_i \| g_i$, and consequently

$$f \| g.$$

Case (ii). $C = R_f$.

(ii). 1. Subcase: $C \neq R_g$. Then $R_f < g_i$ for some $i \in I$ and there exists one (and only one) point $a' \in R$ satisfying

(3) $a' \sim f_i$, $a' < g_i$ ($x \sim y$ means to be in a same node).

Namely, f_i is in the node following C and the chain R_g intersects this node. Hence we have (3). Since $kfI > 1$ we have $f_j \neq a'$ for some $j \in I$; thus $f_j \| a'$ and $f_j \| g_i$ $(i \in I)$; in particular $f_j \| g_j$ i. e. $f \| g$.

(i i) 2. Subcase; $C=R_g$ i. e. $R_f=R_g$. Since $f\neq g, f_i\neq g_i$ for some $i\in I$; thus $f_i\|g_i$, because f_i, g_i are 2 members of the same node of R; the relation $f_i\|g_i$ implies $f\|g$ by definition. The theorem is proved.

6.4.1. Remark. By counterexamples one might prove that both conditions: (i) R is ramified, (i i) f, g are not constant in I are necessary: dropping either of them, one could have $f\leqslant g$ or $f>g$.

6.4.2. Remark. If R is any ramified subset of a ramified set $(R', <)$, then again any two elements f, g of R^I such that f, g satisfy, with respect to R, the conditions of the preceding theorem:

fI is a part of a node of R and $kfI>1$

gI ,, ,, ,, ,, ,, R ,, $kgI>1$

then f, g are incomparable both in R^I and in R'^I.

6.5. The applications of the preceding considerations concern particularly *ramified collections of sets* i. e. collections of sets containing no pair of *interlaced sets* (two sets A, B are interlaced, if both sets $A\setminus B$ and $B\setminus A$ are nonempty).

6.6. We have in particular the following theorem as a particular case of the preceding general theorem (consider I to have just 2 points):

Theorem. *Let F be any family of sets; let $F^{\cdot I2}$ be the set of the cartesian products $x_0\times x_1$ where $x_0, x_1\in F$. If R is any ramified subfamily of F[i. e. $X, Y\in R\Rightarrow X\subseteq Y\vee X\supset Y\vee(X\cap Y=\varnothing)$] such that to every $X\in R$ corresponds an $X'\in R$ satisfying $X\cap X'=\varnothing$ and that X, X' have the same predecessors i. e. supersets in R, then the sets*

$$X\times X'\ (X\in R)$$

are mutually disjoint.

Direct proof. Suppose (3) $(X\cap X')\cap(Y\cap Y')\neq\varnothing$ for some $X, X'\in R$ and some $Y, Y'\in R$.
Then X, Y are comparable; X', Y' as well.
Suppose the case $X\subseteq Y$, hence $X'\subseteq Y$ because X, X' have in R same predecessors. Since $Y'\cap Y=\varnothing$ then Y' is disjoint from X and X', contrary to the hypothetical comparability of X', Y'. The impossibility of other cases is proved in an analogous way.

Corollary. *Let R be a ramified set and N_oR the set of nodes of R each containig at least 2 points. Then the system*

$$\underset{x}{U}[X\times X\setminus diag(X\times X)]\qquad(X\in N_oR)$$

is an antichain in the cartesian square R^2 of R.

6.7. Theorem. *For any square or hypersquare of any tree or ramified set R the chain $\times$ antichain relation holds:*

$$kI>1\Rightarrow kR^I\leqslant k_cR^I\cdot k_{c'}R^I.$$

First of all if R itself satisfies the chain $\times$ antichain relation, then so also R^I. If R does not satisfy this relation then it contains a tree T of the same cardinality (every T which is cofinal with R is such one); T contains (cf. Kurepa) [1] p. 109) a subtree t of the cardinality $kT\,(=kR)$ and such that every isolated node of t contains at least 2 points (cf. the ambiguous tables in Kurepa (1)); according to theorems 6.4 and 6.6 some subtree t_0 of cardinality kt^I of t^I is „normal" i.e. satisfies the chain $\times$ antichain relation; since $t_0 \subseteq R^I$, $kt^I = kR^I$, R^I is normal too.

6.7.1. Remark of course, if for some index set I such that $kI > 1$, *every* subtree of R^I satisfied the chain $\times$ antichain relation (R beinq any ramified set), *the tree hypothesis would hold; and vice versa.*

6.8. For symmetrical graphs we have theorems that read like the ones we formulated and proved for set systems (we define *the cellularity cG of a graph* as the supremum of cardinalities of its antichains). E. g. it is legitimate to substitute „*symmetrical graph G*" instead of „*family F of sets*" in the wording of statements: 3.4, 3.8, 3.9, 3.10, 3.11, 4.10 etc.

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