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ON THE FUNCTIONAL EQUATION: $T_1(t+s)\, T_2(t-s) = T_3(t)\, T_4(s)$

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Throughout this paper $R=\{t, s, u, \ldots\}$ denotes the set of all real numbers, $A=\{T, U, \ldots\}$ a Banach algebra with an identity I, $X=\{x, y, \ldots\}$ a real or complex vector space. In the case of unitary space X, (x, y) will denote the scalar product of vectors x and y. A function $T: R \to A$ is said to be regular if $T(t)$ is a regular element of A for any $t \in R$.

The object of this paper is to study the functional equation

$$T_1(t+s)\; T_2(t-s) = T_3(t)\; T_4(s)$$

where $T_i: R \to A$ are regular functions and all elements $T_i(t)$, $T_j(s)$ commute one with another. This functional equation is generalisation of the well known Cauchy functional equation $T(t+s) = T(t)\; T(s)$, i. e. of the functional equation for one parameter groups (semigroups [1]). Assuming that T_i are sufficiently smooth we prove in Theorem 3 that then T_i are exponential functions of a quadratic polinomial in t. Some generalisations are also made in the case when t and s are replaced by vectors from X. In a special case $T_1 = T_2$ weaker assumptions lead to the same results. Assuming that A is algebra of matrices of finite order and that T_i are measurable in Theorem 5 we prove that then T_i are also continuous. Probably these results can be extended so that measurability implies that T_i are analytical even in the case when T_i are bounded operators on a separable Hilbert space and T_i are weakly measurable.

We follow the terminology of Hille-Philips ([1]). We start with some special cases.

Theorem 1. *Let $T: R \to A$ be a regular and measurable function such that*

(1) $$T(t+s)\, T(t-s) = T^2(t)\, T^2(s)$$

or all t, $s \in R$.

Then, there is one and only one element $T \in A$ such that

$$T(t) = T(0) \exp t^2 T$$

holds for all $t \in R$.

Proof. If we set $t=s=0$ in (1) we get $T^2(0) = T^4(0)$. Since $T(0)$ is regular we find:

(2) $$T^2(0) = I.$$

Now, for $t=0$, (1) and (2) imply

$$T(s)\,T(-s)=T^2(s)$$

which together with the regularity of $T(s)$ imply

$$T(-s)=T(s), \tag{3}$$

i. e. $s\to T(s)$ is an even function. If we replace $t+s$ by t, $t-s$ by s in (1) and if we use (3) we get:

$$T(t)\,T(s)=T(s)\,T(t). \tag{4}$$

Thus the family

$$\{T(t)\,|\,t\in R\}$$

is a family of commuting elements of A. Now set:

$$V(t,\ s)=T(0)\,T(t+s)\,T^{-1}(t)\,T^{-1}(s). \tag{5}$$

Using (1), (2), (4) and (5) one finds:

$$V(t+u,\ s)\,V(t-u,\ s)=V^2(t,\ s)$$

which leads to:

$$V(t+s,\ u)=V(t,\ u)\,V(s,\ u). \tag{6}$$

The measurability of $t\to T(t)$ implies the measurability of $t\to T^{-1}(t)$ and therefore $t\to V(t,\ s)$ is measurable for any s. Thus

$$V(t,\ s)=\exp\ t\,U(s) \tag{7}$$

with $U(s)\in A$ [1]. Obviously $\{U(s)\,|\,s\in R\}$ is a family of commuting elements of A. Now, $V(t,\ s)=V(s,\ t)$ and (6) imply

$$V(u,\ t+s)=V(u,\ t)\,V(u,\ s)$$

which together with (7) leads to

$$\exp\ u\,[U(t+s)-U(t)-U(s)]=I$$

for all $u\in R$ which is possible if and only if

$$U(t+s)=U(t)+U(s) \tag{8}$$

holds for all $t, s\in R$. Since $V(t,s)$ is measurable and

$$U(s)=\lim_{t\to 0}\frac{V(t,s)-I}{t}$$

we conclude that $t\to U(t)$ is measurable. Thus

$$U(t)=2\,t\,T$$

with $T\in A$. Hence

$$V(t,\ s)=\exp 2\,t\,s\,T$$

and therefore by (5)

$$T(t+s)=T(t)\,T(s)\,T(0)\exp 2\,t\,s\,T.$$

For $t=s$ we get:

$$T(2\,t)=T^2(t)\,T(0)\exp 2\,t^2\,T.$$

On the other hand (1) for $t=s$ implies:

$$T(2t)\,T(0)=T^4(t).$$

Hence

$$T^4(t)=T^2(t)\exp 2t^2 T, \text{ i. e.}$$
$$T^2(t)=\exp 2t^2 T.$$

Thus

$$T(t)=T(0)\,T^4\left(\frac{t}{2}\right)=T(0)\exp t^2 T.$$

Since the uniqueness of T is obvious Theorem 1 proved.

Theorem 2. *Suppose that $T, U, V: R \to A$ are regular and measurable functions such that:*

(9) $$T(t+s)\,T(t-s)=U(t)\,V(s), \quad T(0)=V(0)=I$$

holds for all $t, s \in R$.
Then

$$T(t)=\exp\left(\frac{1}{2}t^2 T+tU\right)$$
$$U(t)=\exp(t^2 T+2tU)$$
$$V(t)=\exp t^2 T$$

for all $t \in R$ where T and U are two elements of A which commute one with another.

Proof: For $s=0$ (9) implies $U(t)=T^2(t)$, i. e.

(10) $$T(t+s)\;T(t-s)=T^2(t)\;V(s)$$

holds for all $t, s \in R$. Setting $s=t$ and $s=-t$ in (9) we get:

(11) $$V(-t)=V(t).$$

Replacing $t+s$ by t and $t-s$ by s in (9) and using (11) we have:

$$T(t)\;T(s)=T^2\left(\frac{t+s}{2}\right)V\left(\frac{t-s}{2}\right)=T^2\left(\frac{s+t}{2}\right)V\left(\frac{s-t}{2}\right)=T(s)\;T(t), \text{ i. e.}$$

$T(t)$ and $T(s)$ commute for all $t, s \in R$. From (10) we have

$$V(s)=T^{-2}(t)\;T(t+s)\;T(t-s)$$

from which follows that $T(t)$ and $V(v)$ commute and

$$V(s+u)\,V(s-u)=[T^{-2}(t)\,T(t+s+u)\,T(t-s-u)]\,[T^{-2}(t)\,T(t+s-u)\,T(t-s+u)]=$$
$$=T^{-4}(t)\,[T(t+s+u)\;T(t+s-u)]\,[T(t-s-u)\;T(t-s+u)]=$$
$$=T^{-4}(t)\,[T^2(t+s)\;V(u)]\,[T^2(t-s)\;V(u)]=$$
$$=[T^{-2}(t)\;T(t+s)\;T(t-s)]^2\;V^2(u).$$

Thus

$$V(s+u)\;V(s-u)=V^2(s)\;V^2(u).$$

Since the function $s \to V(s)$ satisfies all conditions of Theorem 1 and $V(0)=I$ we have:

$$V(s)=\exp s^2 T$$

for all $s \in R$ with a unique element $T \in R$. Set

$$W(t) = T(t) \exp\left(-\frac{1}{2} t^2 T\right). \tag{12}$$

Since T and $T(t)$ commute (9) and (12) imply

$$W(t+s)\ W(t-s) = W^2(t)$$

which together with the measurability of $t \to W(t)$ implies:

$$W(t) = \exp t\, U$$

for all $t \in R$ with $U \in A$. Since $T(t)$ and $W(s)$ commute for all t, $s \in R$ we conclude that T and U commute and that:

$$T(t) = \exp \frac{1}{2} t^2 T \exp t\, U = \exp\left(\frac{1}{2} t^2 T + t\, U\right)$$

from which Theorem 2 follows.

Theorem 3. *Suppose that* $T_i : R \to A$ $(i = 1, 2, 3, 4)$ *are functions such that:*

1. $T_j(t)$ *is regular for* $t \in R$ *and* $j = 1, 2, 3, 4$
2. $T_j(t)$ *and* $T_k(s)$ *commute for all* t, $s \in R$ *and* j, $k = 1, 2, 3, 4$ *and*
3. $T_j(t)$ *possesses continuous (strong) second derivative. If*

$$T_1(t+s)\ T_2(t-s) = T_3(t)\ T_4(s) \tag{13}$$

holds for all t, $s \in R$, *then*

$$T_1(t) = T_1(0) \exp(t^2 T + tU_1),\ \ T_2(t) = T_2(0) \exp(t^2 T + tU_2)$$

$$T_3(t) = T_3(0) \exp(2 t^2 T + tU_3),\ \ T_4(t) = T_4(0) \exp(2 t^2 T + tU_4)$$

$$U_3 = U_1 + U_2 \qquad U_4 = U_1 - U_2$$

where T, U_1 *and* U_2 *are elements of* A. *They commute one with another as well as with* $T_i(t)$ *for* $t \in R$ *and* $i = 1, 2, 3, 4$.

Proof: Without loss of generality we assume $T_i(0) = I$ $(i = 1, 2, 3, 4)$. If we take the derivative of (13) with respect to t we get:

$$T_1'(t+s)\, T_2(t-s) + T_1(t+s)\, T_2'(t-s) = T_3'(t)\, T_4(s). \tag{14}$$

Set

$$V_i(t) = T_i^{-1}(t)\, T_i'(t) \qquad (i = 1, 2, 3, 4). \tag{15}$$

From (15), (14) and (13) we get:

$$V_1(t+s) + V_2(t-s) = V_3(t), \tag{16}$$

which implies $V_1'(t) = V_2'(t)$. Thus

$$V_1(t) = 2\, t\, T + U_1$$

$$V_2(t) = 2\, t\, T + U_2$$

$$V_3(t) = 4\, t\, T + U_3,\ \ U_3 = U_1 + U_2$$

with T, U_1, U_2, $U_3 \in A$. Since $V_i(t)$ commutes with $V_j(s)$ for all $t, s \in R$ we conclude that U_1, U_2 and T commute one with another.

Now by (15), $T_1(t)$ satisfies a differential equation:

$$T_1'(t) = (2\,t\,T + U_1)\,T_1(t)$$

which is also satisfied by the function $\exp\,(t^2\,T + tU_1)$.

Hence the function

$$W_1(t) = T_1(t)\ \exp\ (-t^2\,T - t\,U_1)$$

has the property that $W_1'(t) = 0$, i. e. $W_1(t) = W_1(0) = I$ for any $t \in R$. Thus

$$T_1(t) = \exp\ (t^2\,T + t\,U_1).$$

In the same way we find

$$T_2(t) = \exp\ (t^2\,T + t\,U_2)\ \text{ and }\ T_3(t) = \exp\ (2t^2\,T + t\,U_3).$$

From (13) we get $T_4(t) = \exp\ (2t^2\,T + t\,U_4)$ with $U_4 = U_1 - U_2$.

Q. E. D.

Theorem 4. *Let X be a vector space, A a Banach algebra with an identity I, $T_i : X \to A$ $(i = 1, 2, 3, 4)$ regular functions such that:*

$$T_1(x+y)\,T_2(x-y) = T_3(x)\,T_4(y),\ T_i(0) = I \tag{17}$$

holds for all $x, y \in X$.

If $T_i(x)\,T_j(y) = T_j(y)\,T_i(x)$ $(i, j = 1, 2, 3, 4)$ for all x, $y \in X$ and T_i are twice continuously differentiable on R in any direction, i. e. $t \to T_i(t'x)$ is twice continuously differentiable on R for every $x \in X$ then

$$T_1(x) = \exp\ (T(x) + U_1(x)),\ T_2(x) = \exp\ (T(x) + U_2(x))$$

$$T_3(x) = \exp\ (2\,T(x) + U_3(x)),\ T_4(x) = \exp\ (2\,T(x) + U_4(x))$$

$$U_3(x) = U_1(x) + U_2(x),\ U_4(x) = U_1(x) - U_2(x)$$

where T, U_1, U_2, U_3, $U_4 : X \to A$ are such functions that

$$T(x+y) + T(x-y) = 2\,T(x) + 2\,T(y),\ T(tx) = t^2\,T(x),$$

U_i are additive functions and $U_i(tx) = t\,U_i(x)$.

Proof: Set $T_i(t, x) = T_i(tx)$ for $t \in R$ and $x \in X$. If we replace x by tx and y by sx in (17) we get:

$$T_1(t+s, x)\,T_2(t-s, x) = T_3(t, x)\,T_4(s, x)$$

for all $t, s \in R$. Theorem 3 implies:

$$T_1(t, x) = \exp\,[t^2\,T(x) + t\,U_1(x)]$$

$$T_2(t, x) = \exp\,[t^2\,T(x) + t\,U_2(x)]$$

$$T_3(t, x) = \exp\,[2\,t^2\,T(x) + t\,U_3(x)]$$

$$T_4(t, x) = \exp\,[2\,t^2\,T(x) + t\,U_4(x)]$$

with $U_3(x) = U_1(x) + U_2(x)$, $U_4(x) = U_1(x) - U_2(x)$ and $U_1(x)$, $U_2(x)$ and $T(x)$ commute one with another. Now

$$T_1(tx) = \exp[t^2 T(x) + t U_1(x)]$$

implies $T(tx) = t^2 T(x)$ and $U_1(tx) = t U_1(x)$. In the same way we get $U_i(tx) = t U_i(x)$ for $i = 1, 2, 3, 4$. Replacing x by tx and y by ty in (17) we get:

$$T_1(t, x+y) T_2(t, x-y) = T_3(t, x) T_4(t, y)$$

which implies:

$$\exp\{t^2 [T(x+y) + T(x-y) - 2T(x) - 2T(y)] + t[U_1(x+y) + U_2(x-y) - U_3(x) - U_4(y)]\} = I$$

for all $t \in R$. Thus:

$$T(x+y) + T(x-y) = 2T(x) + 2T(y)$$

and

(18) $$U_1(x+y) + U_2(x-y) = U_3(x) + U_4(y).$$

If we replace y by $-y$ in (18) we get:

(19) $$U_1(x-y) + U_2(x+y) = U_3(x) - U_4(y).$$

Adding (18) and (19) we get:

$$U_3(x+y) + U_3(x-y) = 2U_3(x)$$

which implies:

$$U_3(x) + U_3(y) = 2U_3\left(\frac{x+y}{2}\right) = U_3(x+y),$$

i. e. U_3 is additive. If we subtract (19) from (18) we find:

$$U_4(x+y) - U_4(x-y) = 2U_4(y)$$

from which follows that U_4 is additive. But then U_1 and U_2 are also additive.

Q. E. D.

Theorem 5. *Let X be an n-dimensional unitary space, A the algebra of all linear operators defined on X with ranges in X and $T_i : R \to A$ $(i = 1, 2, 3, 4)$ functions such that*

(20) $$T_1(t+s) T_2(t-s) = T_3(t) T_4(s), \ \det T_i(t) \neq 0$$

holds for all $t, s \in R$ and $i = 1, 2, 3$.

If the restrictions of T_1, T_2 and T_3 on an interval $\Delta = [a, b]$, $a < b$, are measurable then T_i $(i = 1, 2, 3, 4)$ are continuous on R.

For the proof we need two lemmas.

Lemma 1. *T_i are measurable on R.*

For $s = \dfrac{b-a}{2}$ (20) gives:

(21) $$T_2\left(t - \frac{b-a}{2}\right) = T_1^{-1}\left(t + \frac{b-a}{2}\right) T_3(t) T_4\left(\frac{b-a}{2}\right)$$

when t runs through $\left[a, \dfrac{1}{2}(a+b)\right] \subseteq \Delta$, then $t + \dfrac{b-a}{2}$ runs over the interval

$\left[\frac{1}{2}(a+b),\, b\right] \subseteq \Delta$. Since T_1 and T_3 are measurable on Δ from (21) we find that T_2 is measurable on the interval $\left[a-\frac{1}{2}(b-a),\, a\right]$.

For $s=-\frac{b-a}{2}$ (20) gives:

$$T_2\left(t+\frac{b-a}{2}\right)=T_1^{-1}\left(t-\frac{b-a}{2}\right)T_3(t)\,T_4\left(\frac{a-b}{2}\right) \tag{22}$$

from which we conclude that T_2 is measurable on the interval $\left[b,\, b+\frac{b-a}{2}\right]$. Thus T_2 is measurable on the interval

$$\left[a-\frac{b-a}{2},\; b+\frac{b-a}{2}\right].$$

Now, (20) implies

$$T_1(t)\;T_2(t-2s)=T_3(t-s)\;T_4(s) \tag{23}$$

which for $s=\pm\frac{b-a}{4}$ leads to:

$$T_1(t)=T_3\left(t-\frac{b-a}{4}\right)\;T_4\left(\frac{b-a}{4}\right)\;T_2^{-1}\left(t-\frac{b-a}{2}\right), \tag{24}$$

$$T_1(t)=T_3\left(t+\frac{b-a}{4}\right)\;T_4\left(-\frac{b-a}{2}\right)\;T_2^{-1}\left(t+\frac{b-a}{2}\right). \tag{25}$$

From (24) we find that T_1 is measurable on $\left[a,\, b+\frac{b-a}{4}\right]$ and from (25) that T_1 is measurable on $\left[a-\frac{b-a}{4},\; b\right]$. Thus T_1 and T_2 are measurable on the interval

$$\Delta'=\left[a-\frac{b-a}{4},\; b+\frac{b-a}{4}\right]. \tag{26}$$

For $s=0$ (20) implies

$$T_3(t)=T_1(t)\;T_2(t)\;T_4^{-1}(0)$$

which implies that T_3 is measurable on Δ'. Thus the measurability of functions T_1, T_2, T_3 on Δ implies the measurability of these functions on Δ'. The way by which Δ' is obtained from Δ enables us to conclude that functions T_1, T_2 and T_3 are measurable on R. But then

$$T_4(s)=T_3^{-1}(0)\;T_1(s)\;T_2(-s)$$

implies the measurability of T_4 on R.

L e m m a 2. *T_i and T_i^{-1} are bounded on every finite interval.*

Since T_i are measurable on $[-1, 1]$, for any $\varepsilon > 0$ the well known Luzin's theorem (Cf. [3]) implies the existence of a perfect set $P \subseteq [-1, 1]$ such that a) $mP > 2-\varepsilon$ and b) T_i are continuous on P. Here mP denotes the Lebesgue measure of P. But then T_i are also continuous on the set

$$Q = P \cap \{-p \mid p \in P\}. \tag{27}$$

Now T_i^{-1} exists and since T_i is continuous on Q so is T_i^{-1}. Thus

$$M = \max_{i=1,2,3,4} \sup_{t \in Q} \{\|T_i(t)\|, \|T_i^{-1}(t)\|\} < +\infty .$$

Since $mQ > 0$, there is a number $c > 0$ with the property that for any $s \in (-c, c)$ there are $t_1(s), t_2(s), t_3(s) \in Q$ such that

$$t_1(s) = t_2(s) + 2s = t_3(s) + s$$

([2], Lemma 1). If in (20) we replace t by $t+s$ we get

$$T_1(t+2s)\ T_2(t) = T_3(t+s)\ T_4(s)$$

from which we find

$$T_4(s) = T_3^{-1}(t+s)\ T_1(t+2s)\ T_2(t). \tag{28}$$

If in (28) we take $s \in (-c, c)$ and $t = t_2(s)$ we get

$$T_4(s) = T_3^{-1}[t_3(s)]\ T_1[t_1(s)]\ T_2[t_2(s)]$$

which implies

$$\|T_4(s)\| \leqslant M^3, \quad s \in (-c, c).$$

Thus T_4 is bounded on the interval $(-c, c)$. If in (20) we replace s by $s-t$ we get:

$$T_1(2t+s)\ T_2(-s) = T_3(t)\ T_4(s+t)$$

which implies:

$$T_3(t) = T_1(2t+s)\ T_2(-s)\ T_4^{-1}(s+t). \tag{29}$$

Now, for $t \in (-c, c)$ there are $s_1(t), s_2(t), s_3(t) \in Q$ such that: $s_1(t) = s_2(t) + 2t = s_3(t) + t$. This and (29) for $s = s_2$ imply

$$T_3(t) = T_1[s_1(t)]\ T_2[-s_2(t)]\ T_4^{-1}[s_3(t)]$$

from which follows

$$\|T_3(t)\| \leqslant M^3, \qquad t \in (-c, c).$$

Thus T_3 and T_4 are bounded on the interval $(-c, c)$. This,

$$T_1(2t) = T_3(t)\ T_4(t)\ T_2^{-1}(0)$$

and

$$T_2(2t) = T_1^{-1}(0)\ T_3(t)\ T_4(-t)$$

imply that all functions T_1, T_2, T_3, T_4 are bounded on the interval $(-c, c)$. In the same way

$$T_2^{-1}(t-s)\, T_1^{-1}(t+s) = T_4^{-1}(s)\, T_3^{-1}(t)$$

implies that T_i^{-1} $(i=1, 2, 3, 4)$ are bounded on $(-c, c)$.

The boundedness of T_i and T_i^{-1} on $(-c, c)$ in the same way as in the case of measurability implies that these functions are bounded on every finite interval.

Proof of theorem 5. From Lemma 1 and Lemma 2 it follows that $t \to (T_i(t)\,x, y)$, $x, y \in X$, is summable on every finite interval. The integral

$$\int_a^b (T_3(t)\,x, y)\,dt$$

defines a linear functional on X for given $y \in X$ and $a, b \in R$. There is therefore a single element $y_{ab} \in X$ such that

$$\int_a^b (T_3(t)\,x, y)\,dt = (x, y_{ab}) \tag{30}$$

holds for all $x \in X$. We assert that the set $Y = \{y_{ab} \mid y \in X, a, b \in R\}$ is dense in X. Indeed let $x_0 \in X$ be orthogonal on Y, *i. e.*

$$\int_a^b (T_3(t)\,x_0, y)\,dt = 0$$

for all $y \in X$ and $a, b \in R$. This implies $(T_3(t)\,x_0, y) = 0$ for all $t \notin S(y)$, where $S(y) \subseteq R$ has the measure zero. If $e_1, e_2, \ldots, e_n$ is a basic set in X, then $(T_3(t)\,x_0, e_k) = 0$ $(k = 1, 2, \ldots, n)$ for all $t \notin S = \bigcup_1^n S(e_k)$. Thus $T_3(t)\,x_0 = 0$ for almost all $t \in R$. Since T_3 is regular we find $x_0 = 0$, i. e. Y is dense in X. Thus for the continuity of T_4 it is sufficient to prove that $(T_4(s)\,x, y_{ab})$ is continuous for all $x \in X$ and $y_{ab} \in Y$. Using (30) and (20) we have:

$$(T_4(s)\,x, y_{ab}) = \int_a^b (T_3(t)\; T_4(s)\,x, y)\,dt = \int_a^b (T_1(t+s)\,T_2(t-s)\,x, y)\,dt$$
$$= \int_{a+s}^{b+s} (T_1(t)\; T_2(t-2s)\; x,\; y)\; dt.$$

Suppose that $s_k \to s_0$. Then:

$$\left| \int_{a+s_k}^{b+s_k} (T_1(t)\,T_2(t-2s_k)\,x, y)\,dt - \int_{a+s_0}^{b+s_0} (T_1(t)\; T_2(t-2s_0)\; x,\; y)\; dt \right|$$
$$\leqslant \left| \int_{a+s_k}^{b+s_k} (T_1(t)\; T_2(t-2s_k)\,x, y)\,dt - \int_{a+s_0}^{b+s_0} (T_1(t)\; T_2(t-2s_k)\; x,\; y)\; dt \right| +$$
$$+ \left| \int_{a+s_0}^{b+s_0} (T_1(t)\,[T_2(t-2s_k) - T_2(t-2s_0)]\; x, y)\,dt \right|$$
$$\leqslant 2\,M_{ab}^2\, \|x\| \cdot \|y\|\; |s_k - s_0| +$$
$$+ \left| \int_{a+s_0}^{b+s_0} (T_1(t)\,[T_2(t-2s_k) - T_2(t-2s_0)]\; x, y)\,dt \right|$$

where M_{ab} is a suitable constant such that $\| T_1(t) \| \leqslant M_{ab}$, $\| T_2(t) \| \leqslant M_{ab}$ for all $t \in (-c, c)$ with $c > 0$ sufficiently large so that all intervals $(a+s_0, a+s_k)$, $(b+s_0, b+s_k)$ belong to $(-c, c)$.

In order to prove that the last integral tends to zero as $s_k \to s_0$ we take an orthonormal basic set $e_1, \dots, e_n$ in X and we replace x by e_j and y by e_p. We have:

$$\left| \int_{a+s_0}^{b+s_0} (T_1(t) [T_2(t-2s_k) - T_2(t-2s_0)] e_j, e_p) \, dt \right| =$$

$$= \left| \sum_{q=1}^{n} \int_{a+s_0}^{b+s_0} (T_1(t) e_q, e_p) ([T_2(t-2s_k) - T_2(t-2s_0)] e_j, e_q) \, dt \right|$$

$$\leqslant M'_{ab} \sum_{q=1}^{n} \int_{a+s_0}^{b+s_0} | (T_2(t-2s_k) e_j, e_q) - (T_2(t-2s_0) e_j, e_q) | \, dt$$

with a suitable constant M'_{ab}. Since every integral in this sum tends to zero as $s_k \to s_0$ ([4], pp. 163—164) we find

$$\int_{a+s_0}^{b+s_0} (T_1(t) [T_2(t-2s_k) - T_2(t-2s_0)] e_j, e_p) \to 0.$$

Thus $s_k \to s_0$ implies $(T_4(s_k) x, y_{ab}) \to (T_4(s_0) x, y_{ab})$ which proves the continuity of T_4 on R.

From $T_2^{-1}(t-s) T_1^{-1}(t+s) = T_4^{-1}(s) \; T_3^{-1}(t)$ in the similar way we find that T_3^{-1} is continuous on R. Thus T_3 is also continuous on R. If we replace $t+s$ by t and $t-s$ by s in (20) we get:

$$T_1(t) \; T_2(s) = T_3\left(\frac{t+s}{2}\right) \; T_4\left(\frac{t-s}{4}\right).$$

Hence

$$T_1(t) = T_3\left(\frac{t}{2}\right) \; T_4\left(\frac{t}{2}\right) \; T_2^{-1}(0) \qquad \text{and} \qquad T_2(s) = T_1^{-1}(0) \; T_3\left(\frac{s}{2}\right) \; T_4\left(-\frac{s}{2}\right)$$

from which we conclude that T_1 and T_2 are also continuous on R.

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