PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 2 (16), 1962, p. 93—98.

THE MUTUAL INCLUSION OF $S^{\alpha,\beta}$ METHODS OF SUMMATION

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(Received 21. I 1963)

1. In [3] the author studied $S^{\alpha,\beta}$ methods for the evaluation of sequences and series which were introduced by *V. Vučković* in [4].

For each $0 \leqslant \nu \leqslant n$, $n=1, 2, \ldots$, let the polynomials $\sigma_\nu^n(\alpha)$ and $\tau_\nu^n(\alpha)$ be defined by

$$\prod_{\nu=0}^{n-1}(x+\alpha+\nu)=\sum_{\nu=0}^{n}\sigma_\nu^n(\alpha)\,x^\nu$$

$$x^n=(-1)^n\left\{\tau_0^n(\alpha)+\sum_{\nu=1}^{n}(-1)^\nu\,\tau_\nu^n(\alpha)\prod_{k=0}^{\nu-1}(x+\alpha+k)\right\}$$

and for $\nu > n$ and $\nu=-1, -2, \ldots$, let $\sigma_\nu^n(\alpha)=\tau_\nu^n(\alpha)=0$.

The sequence s_n is $S^{\alpha,\beta}$ summable to s if

$$S^{\alpha,\beta}\{s_n\}\overset{\text{def}}{=}\frac{1}{\prod\limits_{\nu=0}^{n-1}(\alpha+\beta+\nu)}\sum_{\nu=0}^{n}\sigma_\nu^n(\alpha)\,\beta^\nu\,s_\nu\longrightarrow s,\ n\longrightarrow\infty \tag{1}$$

$$(\alpha > -1,\ \beta > 0,\ \alpha+\beta \neq 0).$$

Karamata-Stirling [1], [4] and [5], *Vučković* σ^α [4] and *Lototsky* [2] summability are special case of $S^{\alpha,\beta}$ methods.

In [3] the author proved

Theorem 1. *For $\alpha > -1$, $\beta > 0$, $\alpha+\beta > 0$ and every $\varepsilon > 0$*

$$S^{\alpha,\beta}\subset S^{\alpha+\varepsilon,\beta} \tag{2}$$

i. e. the $S^{\alpha+\varepsilon,\beta}$ methods include $S^{\alpha,\beta}$ methods.

In this paper we shall prove a theorem on the inclusion of the methods $S^{\alpha,\beta}$ in regard to the parameter β. This is

Theorem 2. *For $\alpha \geqslant 0$, $\beta > 0$ and every θ, $0 < \theta < 1$.*

$$S^{\alpha,\beta}\subset S^{\alpha,\theta\beta} \tag{3}$$

Theorems 1 and 2 could be superposed to give

Theorem 3. *For* $\alpha \geqslant 0$, $\beta > 0$ *and every* $\varepsilon > 0$ *and* $0 < \theta < 1$

$$S^{\alpha,\beta} \subset S^{\alpha+\varepsilon,\theta\beta} \tag{4}$$

Namely, first by Theorem 1 we have

$$S^{\alpha,\beta} \subset S^{\alpha+\varepsilon,\beta}$$

then, by Theorem 2

$$S^{\alpha+\varepsilon,\beta} \subset S^{\alpha+\varepsilon,\theta\beta}$$

These inclusions give (4) under the conditions of Theorem 2 on α and β (which include the conditions of Theorem 1).

It is interesting to note that Theorem 1 contains as a special case the Theorem 5.2 of [4] (for $\beta = 1$), and that Theorem 2 contains as a special case the Theorem 1 of [5] (for $\alpha = 0$). But their union (i. e. Theorem 3) does not contain the Theorem 5.2 of [4]; the reasons for these are the conditions of Theorem 2, which had to be incorpored in Theorem 3.

2. We split the proof of Theorem 2 in two lemmas

Lemma 1. *For every* $\alpha \geqslant 0$, $\beta > 0$, $0 < \theta < 1$ *from* $S^{\alpha,\beta}\{s_n\} \to s$, $n \to \infty$ *it follows* $S^{\alpha,\theta\beta}\{s_n\} \to s$, $n \to \infty$.

Lemma 2. *The sequence*

$$s_n = \int_0^\infty e^{-t}(-tx)^{n+s}\,dt, \quad (x > 0,\ s \text{ is natur. numb.})$$

for $(k\beta\log 2)^{-1} < x < (\beta\log 2)^{-1}$, $k > 1$ *is* $S^{\alpha,\beta}$ $(\alpha \geqslant 0,\ \beta > 0)$ *summable to the value* 0, *but is not* $S^{\alpha,k\beta}$ *summable.*

Proof of lemma 1. Suppose that

$$S^{\alpha,\beta}\{s_n\} = \frac{1}{\prod\limits_{\nu=0}^{n-1}(\alpha+\beta+\nu)} \sum_{\nu=0}^{n} \sigma_\nu^n(\alpha)\,\beta^\nu s_\nu \to s,\ n \to \infty \quad (\alpha \geqslant 0,\ \beta > 0). \tag{5}$$

Using (5) and Theorem 2.2 in [4] it follows

$$\beta^\nu s_\nu = (-1)^\nu \sum_{i=0}^{\nu} (-1)^i \tau_i^\nu(\alpha)\, S^{\alpha,\beta}\{s_i\} \prod_{k=1}^{i}(\alpha+\beta+k-1)$$

$\left(\text{for } \prod\limits_{k=1}^{0}(\alpha+\beta+k-1) \text{ one needs to take } 1\right)$ and hence for $0 < \theta < 1$

$$S^{\alpha,\theta\beta}\{s_n\} = \frac{1}{\prod\limits_{\nu=0}^{n-1}(\alpha+\theta\beta+\nu)} \sum_{\nu=0}^{n} \sigma_\nu^n(\alpha)\,(-\theta)^\nu \sum_{i=0}^{\nu}(-1)^i \tau_i^\nu(\alpha)\, S^{\alpha,\beta}\{s_i\} \prod_{k=1}^{i}(\alpha+\beta+k-1).$$

Changing the order of summation we have

$$S^{\alpha,\theta\beta}\{s_n\} = \frac{1}{\prod\limits_{\nu=0}^{n-1}(\alpha+\theta\beta+\nu)} \sum_{\nu=0}^{n} S^{\alpha,\beta}\{s_\nu\} \prod_{k=1}^{\nu}(\alpha+\beta+k-1)\, W_\nu^n \tag{6}$$

where

$$W_\nu^n = (-1)^\nu \sum_{i=\nu}^{n} (-\theta)^i \sigma_i^n(\alpha)\, \tau_\nu^i(\alpha) = \theta^\nu \sum_{i=0}^{n-\nu} (-1)^i \sigma_{\nu+i}^n(\alpha)\, \tau_\nu^{\nu+i}(\alpha)\,\theta^i \tag{7}$$

We shall show that the triangular matrix $((p_{n\nu}))$ with

(8) $$p_{n\nu}=\frac{\prod_{k=1}^{\nu}(\alpha+\beta+k-1)}{\prod_{\nu=0}^{n-1}(\alpha+\theta\beta+\nu)}W_\nu^n$$

is regular. The fact that $\sum_{\nu=0}^{n} p_{n\nu}=1$ is obvious. From (7) and the recurrence formulae for $\sigma_\nu^n(\alpha)$ and $\tau_\nu^n(\alpha)$ in [4] we have the recurrence formula

(9) $$W_\nu^n=(n-1+\alpha-\theta\alpha-\theta\nu)\,W_\nu^{n-1}+\theta\,W_{\nu-1}^{n-1},\quad n=2,3,\ldots;\ \nu=1,2,\ldots$$

We have, for instance,

$W_0^0=1$

$W_0^1=(1-\theta)\,\alpha,\qquad W_1^1=\theta$

$W_0^2=(1-\theta)\,(1+\alpha-\theta\alpha)\,\alpha,\quad W_1^2=(1-\theta)\,\theta\,(1+2\alpha),\qquad W_2^2=\theta^2$

$W_0^3=(1-\theta)\,(2+\alpha-\theta\alpha)\,(1+\alpha-\theta\alpha)\,\alpha,$

$W_1^3=(1-\theta)\,\theta\,\{[2+\alpha-\theta\,(1+\alpha)]\,(1+2\alpha)+(1+\alpha-\theta\alpha)\,\alpha\},\quad W_2^3=3\,(1-\theta)\,\theta^2\,(1+\alpha),$

$W_3^3=\theta^3$

and generally

(10) $$W_0^n=\prod_{\nu=0}^{n-1}(\nu+\alpha-\theta\alpha),\ \ldots,\quad W_n^n=\theta^n$$

For $\alpha\geqslant 0$ the matrix $((p_{n\nu}))$ and W_ν^n are positive. Therefore, from (9) one gets

(11) $$W_\nu^n\leqslant(n-1+\alpha-\theta\alpha)\,W_\nu^{n-1}+W_{\nu-1}^{n-1},\quad n=2,3,\ldots;\ \nu=1,2,\ldots$$

i. e.

(12)
$$\begin{aligned}
&W_1^1\leqslant 1, \qquad W_1^2\leqslant(1+\alpha-\theta\alpha)\,W_1^1+(\alpha-\theta\alpha)\\
&W_1^3\leqslant(2+\alpha-\theta\alpha)\,W_1^2+(\alpha-\theta\alpha)\,(1+\alpha-\theta\alpha)\\
&W_1^{n-1}\leqslant(n-2+\alpha-\theta\alpha)\,W_1^{n-2}+(\alpha-\theta\alpha)\,(1+\alpha-\theta\alpha)\cdots(n-3+\alpha-\theta\alpha)\\
&W_1^n\leqslant(n-1+\alpha-\theta\alpha)\,W_1^{n-1}+(\alpha-\theta\alpha)\,(1+\alpha-\theta\alpha)\cdots(n-2+\alpha-\theta\alpha).
\end{aligned}$$

Multiplying in (12) second inequality by $\frac{1}{1+\alpha-\theta\alpha}$, third by $\frac{1}{(1+\alpha-\theta\alpha)\,(2+\alpha-\theta\alpha)}$, ..., last by $\frac{1}{\prod_{\nu=1}^{n-1}(\nu+\alpha-\theta\alpha)}$ and then summing all of them we obtain

(13) $$W_1^n\leqslant\left\{1+\frac{\alpha-\theta\alpha}{1+\alpha-\theta\alpha}+\frac{\alpha-\theta\alpha}{2+\alpha-\theta\alpha}+\cdots+\frac{\alpha-\theta\alpha}{n-1+\alpha-\theta\alpha}\right\}\prod_{\nu=1}^{n-1}(\nu+\alpha-\theta\alpha)=I_{n-1}(\alpha,\theta)\prod_{\nu=1}^{n-1}(\nu+\alpha-\theta\alpha)$$

where $I_{n-1}(\alpha,\theta)=\sum_{\nu=0}^{n-1}\frac{1}{\nu+\alpha-\theta\alpha}$. We shall now prove by induction that for every fixed ν, $0\leq\nu\leq n$

$$W_\nu^n\leq\{I_{n-1}(\alpha,\theta)\}^\nu\prod_{\nu=0}^{n-1}(\nu+\alpha-\theta\alpha),\qquad n=2,3,\ldots \tag{14}$$

For $\nu=0$ and $\nu=1$, (10) and (13) show us that (14) holds. Let be (14) correct for a $\nu<n$. Then (11) gives

$$W_{\nu+1}^n\leq(n-1+\alpha-\theta\alpha)\,W_{\nu+1}^{n-1}+W_\nu^{n-1}$$

i. e.

$$W_{\nu+1}^n\leq(n-1+\alpha-\theta\alpha)\,W_{\nu+1}^{n-1}+\{I_{n-2}(\alpha,\theta)\}^\nu\prod_{\nu=0}^{n-2}(\nu+\alpha-\theta\alpha)$$

Multiplying the last inequality by $\left\{\prod_{\nu=0}^{n-1}(\nu+\alpha-\theta\alpha)\right\}^{-1}$ and summing over $\nu+2\leq n\leq N$ we get

$$\frac{W_{\nu+1}^N}{\prod_{\nu=0}^{N-1}(\nu+\alpha-\theta\alpha)}\leq\frac{1}{\prod_{k=0}^{\nu}(k+\alpha-\theta\alpha)}+\frac{1}{\nu+1+\alpha-\theta\alpha}\{I_\nu(\alpha,\theta)\}^\nu+$$

$$+\frac{1}{\nu+2+\alpha-\theta\alpha}\{I_{\nu+1}(\alpha,\theta)\}^\nu+\cdots+\frac{1}{N-1+\alpha-\theta\alpha}\{I_{N-2}(\alpha,\theta)\}^\nu\leq$$

$$\leq\{I_{N-2}(\alpha,\theta\}^\nu\{(\nu+\alpha-\theta\alpha)^{-1}+(\nu+1+\alpha-\theta\alpha)^{-1}+\cdots+(N-1+\alpha-\theta\alpha)^{-1}\}\leq$$

$$\leq\{I_{N-1}(\alpha,\theta)\}^\nu I_{N-1}(\alpha,\theta)=\{I_{N-1}(\alpha,\theta)\}^{\nu+1}$$

q. e. d. Therefore (8), (13) and (14) give for every fixed ν

$$p_{n\nu}=O\left\{\frac{(n-1)!\;n^{\alpha-\theta\alpha}\log^\nu n}{(n-1)!\;n^{\alpha+\theta\beta}}\right\}=o(1),\quad n\to\infty$$

and all conditions for regularity of the matrix $((p_{n\nu}))$ are satisfied. Therefore (5), (6) and (8) prove lemma 1.

Proof of lemma 2. The $S^{\alpha,\beta}\,(\alpha\geq0)$ transform of our sequence is

$$S^{\alpha,\beta}\left\{\int_0^\infty e^{-t}(-tx)^{n+s}\,dt\right\}=\frac{1}{\prod_{\nu=0}^{n-1}(\alpha+\beta+\nu)}\sum_{\nu=0}^{n}\sigma_\nu^n(\alpha)\,\beta^\nu\int_0^\infty e^{-t}(-tx)^{\nu+s}\,dt=$$

$$=\frac{1}{\beta^s\prod_{\nu=0}^{n-1}(\alpha+\beta+\nu)}\int_0^\infty e^{-t}(-tx\beta)^s(\alpha-tx\beta)(\alpha-tx\beta+1)\cdots(\alpha-tx\beta+n-1)\,dt=$$

$$=\frac{(-1)^s e^{-\frac{\alpha}{x\beta}}}{\beta^s\prod_{\nu=0}^{n-1}(\alpha+\beta+\nu)}\int_{-\alpha}^\infty e^{-\frac{T}{x\beta}}(T+\alpha)^s\;(-T)\;(-T+1)\cdots(-T+n-1)\;d\left(\frac{T}{x\beta}\right)$$

$$=\frac{(-1)^{n+s}e^{-\frac{\alpha}{x\beta}}}{x\beta^{s+1}\prod_{\nu=0}^{n-1}(\alpha+\beta+\nu)}\int_{-\alpha}^\infty e^{-\frac{T}{x\beta}}(T+\alpha)^s\,T(T-1,\cdots(T-n+1)\,dT.$$

We split the last integral into three parts:

$$(15)\qquad S^{\alpha,\beta}\left\{\int_0^\infty e^{-t}(-tx)^{n+s}\,dt\right\}=A_1\left\{\int_{-\alpha}^{0}(\quad)\,dT+\int_0^{n-1}(\quad)\,dT+\int_{n-1}^{\infty}(\quad)\,dT\right\}$$

where

$$A_1=(-1)^{n+s}\,e^{-\frac{\alpha}{x\beta}}\left\{x\beta^{s+1}\prod_{\nu=1}^{n-1}(\alpha+\beta+\nu)\right\}^{-1}.$$

For the first

$$\left|\int_{-\alpha}^{0}(\quad)\,dT\right|\leqslant\int_0^{\alpha}e^{\frac{t}{x\beta}}(t+\alpha)^s\,t\,(t+1)\cdots(t+n-1)\,dt$$

$$\leqslant e^{\frac{\alpha}{x\beta}}(2\alpha)^s\,\alpha\,(\alpha+1)\cdots(\alpha+n-1)\,\alpha$$

and for the second

$$\left|\int_0^{n-1}(\quad)\,dT\right|\leqslant\sum_{k=1}^{n-1}\int_{k-1}^{k}T(T+\alpha)^s\,e^{-\frac{T}{x\beta}}\left|(T-1)(T-2)\cdots(T-n+1)\right|dT.$$

But for $k-1\leqslant T\leqslant k$

$$\left|(T-1)\ (T-2)\cdots(T-n+1)\right|\leqslant(n-1)!$$

and fol'ows

$$\left|\int_0^{n-1}(\quad)\,dT\right|\leqslant(n-1)!\int_0^{n-1}T(T+\alpha)^s\,e^{-\frac{T}{x\beta}}\,dT=$$

$$=(n-1)!\ O(1),\ n\to\infty.$$

Therefore

$$(16)\qquad \left|A_1\int_{-\alpha}^{0}(\quad)\,dT\right|\to0,\ n\to\infty;\qquad \left|A_1\int_0^{n-1}(\quad)\,dT\right|\to0,\ n\to\infty$$

since $\alpha\geqslant0$, $\beta>0$. For the third integral we have ($[u]$ is the smallest integer not smaller then u)

$$\left|\int_{n-1}^{0}(\quad)\,dT\right|\leqslant e^{-\frac{n-1}{x\beta}}\int_0^\infty e^{-\frac{t}{x\beta}}(t+n-1+[\alpha])^s\,(t+n-1)\,(t+n-2)\cdots t\,dt=$$

$$=e^{-\frac{n-1}{x\beta}}\sum_{k=0}^{\infty}\int_k^{k+1}e^{-\frac{t}{x\beta}}\,(t+n-1+[\alpha])^s\,(t+n-1)\,(t+n-2)\cdots t\,dt\leqslant$$

$$\leqslant e^{-\frac{n-1}{x\beta}}\sum_{k=0}^{\infty}e^{-\frac{k}{x\beta}}\,(n+k+[\alpha])^s\,(n+k)\,(n+k-1)\cdots(k+1)\leqslant$$

$$\leqslant e^{-\frac{n-1}{x\beta}}\sum_{k=0}^{\infty}e^{-\frac{k}{x\beta}}\,(n+k+[\alpha]+s)\,(n+k+[\alpha]+s-1)\cdots(n+k)\,(n+$$

$$+k-1)\cdots(k+1)=e^{-\frac{n-1}{x\beta}}\,(n+s+[\alpha])!\sum_{k=0}^{\infty}\binom{n+k+s+[\alpha]}{n+s+[\alpha]}e^{-\frac{k}{x\beta}}=$$

$$(17) \qquad = \frac{e^{-\frac{n-1}{x\beta}}(n+s+[\alpha])!}{\left(1-e^{-\frac{1}{x\beta}}\right)^{n+s+[\alpha]+1}} = O\,(1)\,\frac{(n+s+[\alpha])!}{\left(e^{\frac{1}{x\beta}}-1\right)^{n}}, \quad n\to\infty.$$

We see from (15), (16) and (17) that the sequence

$$\int_0^\infty e^{-t}(-xt)^{n+s}\,dt \quad \text{is} \quad S^{\alpha,\beta}\;(\alpha\geqslant 0,\ \beta>0)$$

summable to the value 0 for $0<x\beta\log 2<1$.

Take now the case $x\beta\log 2>1$. We still have the estimate (16), but for the third integral we have now

$$\int_{n-1}^{\infty}(\quad)\,dT \geqslant e^{-\frac{n-1}{x\beta}}(n-1+\alpha)^s\int_0^\infty e^{-\frac{t}{x\beta}}(t+n-1)(t+n-2)\cdots t\,dt\geqslant$$

$$\geqslant e^{-\frac{n-1}{x\beta}}(n-1+\alpha)^s\sum_{k=0}^{\infty}e^{-\frac{k+1}{x\beta}}(n-1+k)(n-2+k)\cdots k=$$

$$=e^{-\frac{n-1}{x\beta}}(n-1+\alpha)^s\sum_{k=0}^{\infty}e^{-\frac{k+2}{x\beta}}(n+k)(n-1+k)\cdots(k+1)=$$

$$=e^{-\frac{n+1}{x\beta}}n!\;(n-1+\alpha)^s\sum_{k=0}^{\infty}\binom{n+k}{n}e^{-\frac{k}{x\beta}}=$$

$$=\frac{e^{-\frac{n+1}{x\beta}}n!\,(n-1+\alpha)^s}{\left(1-e^{-\frac{1}{x\beta}}\right)^{n+1}}=O\,(1)\,\frac{n!\,n^s}{\left(e^{\frac{1}{x\beta}}-1\right)^{n}},\quad n\to\infty$$

and

$$\left|A_1\int_{n-1}^{\infty}(\quad)\,dT\right|\to\infty,\quad n\to\infty$$

for $x\beta\log 2>1$.

Therefore for $(k\beta\log 2)^{-1}<x<(\beta\log 2)^{-1}$, $k>1$, the given sequence is $S^{\alpha,\beta}\,(\alpha\geqslant 0,\ \beta>0)$ summable but is not $S^{\alpha,k\beta}\,(\alpha\geqslant 0,\ \beta>0)$ summable. This proves the lemma 2.

Lemmas 1 and 2 give *Theorem* 2.

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