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ON A THEOREM OF TITCHMARSH

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1. E. C. Titchmarch [5] proved the following

Theorem A. Let $0<\alpha\leqslant 1$, $1<p\leqslant 2$ and $h>0$. If $f(x)$ is even and belongs to Lip (α, p), i. e.

$$\int_0^\infty |f(x+h)-f(x)|^p\,dx=O(h^{\alpha p}),$$

then the Fourier cosine transform of $f(x)$ belongs to L^β, for

$$p/(p+\alpha p-1)<\beta<p/(p-1).$$

The theorem analogous to this on Fourier series is the following

Theorem B. Let $f(x)$ be L-integrable in $(0, 2\pi)$ and periodic outside with period 2π and let

$$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^\infty(a_n\cos nx+b_n\sin nx). \tag{1}$$

If $f(x)\varepsilon$ Lip (α, p), i. e.

$$\int_0^{2\pi}|f(x+h)-f(x)|^p dx=O(h^{\alpha p}),$$

where $0<\alpha\leqslant 1$, $1<p\leqslant 2$, $h>0$, then

$$\sum_{n=1}^\infty(|a_n|^\beta+|b_n|^\beta)<\infty, \tag{2}$$

for $\beta>p/(p+\alpha p-1)$. For $\beta=p/(p+\alpha p-1)$, (2) need not hold; sec O. Szàsz [4]. The case for $p=2$ was proved by S. Berstein [1] and O. Szàsz [3].

If we take $\alpha p>1$, then $p/(p+\alpha p-1)<1$ and hence Theorem B implies the absolute convergence of the Fourier series of $f(x)$ in this case. However, Min-Teh Cheng [2] has proved the following

Theorem C. If $\varepsilon>0$, $1<p\leqslant 2$, $h>0$ and

$$\int_0^{2\pi}|f(x+h)-f(x)|^p\,dx=O(h(\log h^{-1})^{-p-\varepsilon}),$$

then the Fourier series of $f(x)$ converges absolutely. For $\varepsilon=0$, this is no longer true. In fact, this theorem has been proved by Min-Teh Cheng in the following more general form:

Theorem D. If $0<\alpha\leqslant 1$, $1<p\leqslant 2$, $h>0$ and

$$\int_0^{2\pi} |f(x+h)-f(x)|^p\,dx = O\left(h(\log h^{-1})^{-1-\alpha p}\right), \tag{3}$$

then

$$\sum_{n=2}^{\infty} (|a_n|+|b_n|)\log^T n<\infty, \tag{4}$$

for $T<\alpha+p^{-1}-1$. Moreover, (4) may not hold for $T=\alpha-p^{-1}-1$.

In order to prove Theorem D, Min-Teh Cheng has first established an inequality for Lip (α, p) classes corresponding to Hausdorff — Young inequality [7; Vol. II, p. 101] and with the help of the inequality he has also given an alternative proof of Theorem B. But, as suggested by A. Zygmund [7; Vol. I, p. 251], Theorem B can be proved directly with the help of Hausdorff — Young inequality. One of the objects of this paper is to provide an alternative proof for Theorem D, based merely on Hausdorff — Young inequality, by suitably amending the classical method of S. Bernstein and without appealing to the inequality for Lip (α, p) classes obtained by Min-Teh Cheng. Thus the proof which we shall have is comparatively short and more direct. We shall also generalize Theorem D to the following

Theorem 1. If $0<\alpha\leqslant 1$, $1<p\leqslant 2$, $h>0$ and

$$\int_0^{2\pi} |f(x+h)-f(x)|^p\,dx = O\left(h^{\delta}(\log h^{-1})^{-1-\alpha p}\right), \tag{5}$$

where $\delta=1+p(1-\beta)/\beta$, then

$$\sum_{n=2}^{\infty} (|a_n|^{\beta}+|b_n|^{\beta})\log^T n<\infty, \tag{6}$$

for $\beta>p(T+1)/(1+\alpha p)$. For $p=p(T+1)/(1+\alpha p)$, (6) does not hold. We also state that Theorem C is contained in the following more general result already proved by the author in [6]:

Theorem 2. If for $h>0$,

$$l_1(h)=\log(e+h^{-1}),$$

$$l_2(h)=\log\log(e^e+h^{-1}),\ \text{etc.},$$

and if for certain $\varepsilon>0$,

$$\left(\int_0^{2\pi} |f(x+h)-f(x)|^p\,dx\right)^{1/p} = O\left(\frac{h^{\alpha}}{(l_1(h)\,l_2(h)\ldots l_k^{1+\varepsilon}(h))^{\gamma}}\right),$$

where $0<\alpha\leqslant 1$ and $\gamma=(p+\alpha p-1)/p$, then (2) holds for $\beta=p/(p+\alpha p-1)$.

2. In Theorem D, we may naturally ask: what condition must $f(x)$ satisfy in order that (4) may hold for $T=\alpha+p^{-1}-1$? In this connection, we shall prove the following

Theorem 3. If $0<\alpha\leqslant 1$, $1<p\leqslant 2$, $h>0$ and

$$\int_0^{2\pi} |f(x+h)-f(x)|^p\,dx=0\left(h(\log h^{-1})^{-(1+\alpha p)}(\log\log h^{-1})^{-(1+\varepsilon)p}\right), \tag{7}$$

then (4) holds for $T=\alpha+p^{-1}-1$.

3. Alternative proof of Theorem D

From (1), it follows that

$$f(x+h)-f(x-h)\sim 2\sum_{n=1}^{\infty}(-a_n\sin nx+b_n\cos nx)\sin nh.$$

Therefore, by Hausdorff — Young inequality, we get

$$\left(\sum_{n=1}^{\infty}(2\rho_n|\sin nh|)^{p'}\right)^{1/p'}\leqslant\left(\frac{1}{2\pi}\int_0^{2\pi}|f(x+h)-f(x-h)|^p\,dx\right)^{1/p},$$

where $\rho_n^{p'}=|a_n|^{p'}+|b_n|^{p'}$ and p' is given by $\dfrac{1}{p}+\dfrac{1}{p'}=1$.

Putting $n=\dfrac{\pi}{2N}$, we get from (3)

$$\left(\sum_{n=1}^{N}\left(\rho_n\left|\sin\frac{n\pi}{2N}\right|\right)^{p'}\right)^{1/p'}\leqslant\left(\sum_{n=1}^{\infty}\left(\rho_n\left|\sin\frac{n\pi}{2N}\right|\right)^{p'}\right)^{1/p'}$$

$$\leqslant\frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(x+\frac{\pi}{2N}\right)-f\left(x-\frac{\pi}{2N}\right)\right|^p dx\right)^{1/p}$$

$$=O\left(\frac{N^{-1/p}}{(\log(N/\pi))^{(1+\alpha p)/p}}\right).$$

Taking $N=2^\nu$ and taking into account only the terms with indices exceeding $\dfrac{1}{2}N$, we obtain

$$\sum_{n=2^{\nu-1}+1}^{2^\nu}\left(\rho_n\left|\sin\frac{n\pi}{2^{\nu+1}}\right|\right)^{p'}=O\left(\frac{2^{-\nu p'/p}}{\left(\log\dfrac{2^\nu}{\pi}\right)^{(1+\alpha p)p'/p}}\right).$$

Since $\sin\dfrac{n\pi}{2^{\nu+1}}>\dfrac{1}{\sqrt{2}}$, for $2^{\nu-1}<n\leqslant 2^\nu$, it follows that

$$\sum_{n=2^{\nu-1}+1}^{2^\nu}\rho_n^{p'}=O\left(\frac{2^{-\nu p'/p}}{\left(\log\dfrac{2^\nu}{\pi}\right)^{(1+\alpha p)p'/p}}\right).$$

Now, by Hölder's inequality,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \leqslant \Big(\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{p'}\Big)^{1/p'} \Big(\sum_{n=2^{\nu-1}+1}^{2^\nu} 1\Big)^{1-1/p'}$$

$$= O\left(\frac{2^{-\nu/p}}{\left(\log \frac{2^\nu}{\pi}\right)^{(1+\alpha p)/p}} \cdot 2^{\nu/p}\right) = O\left(\frac{1}{\left(\log \frac{2^\nu}{\pi}\right)^{(1+\alpha p)/p}}\right);$$

and hence

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \log^T n \leqslant \log^T 2^\nu \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n$$

$$= O\left(\frac{\log^T 2^\nu}{\left(\log \frac{2^\nu}{\pi}\right)^{(1+\alpha p)/p}}\right).$$

Finally,

$$\sum_{n=2}^{\infty} \rho_n \log^T n = \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \log^T n = O\left(\sum_{\nu=1}^{\infty} \frac{\log^T 2^\nu}{\left(\log \frac{2^\nu}{\pi}\right)^{(1+\alpha p)/p}}\right)$$

$$= O\Big(\sum_{\nu=1}^{\infty} \nu^{T-(1+\alpha p)/p}\Big) = O(1),$$

for $T<\alpha+p^{-1}-1$.

This completes the proof of the theorem.

Proof of Theorem 1. As in the proof of Theorem D, we obtain from (5)

$$\sum_{n=2^{\nu-1}+p}^{2^\nu} \rho_n^{p'} = O\left(\frac{2^{-\nu\delta p'/p}}{\left(\log \frac{2^\nu}{\pi}\right)^{(1+\alpha p)p'/p}}\right).$$

Now, by Hölder's inequality, we get

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^\beta \leqslant \Big(\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{p'}\Big)^{\beta/p'} \Big(\sum_{n=2^{\nu-1}+1}^{2^\nu} 1\Big)^{1-\beta/p'}$$

$$= O\left(\frac{2^{-\nu\delta\beta/p}}{\left(\log \frac{2^\nu}{\pi}\right)^{\beta(1+\alpha p)/p}} \cdot 2^{\nu(1-\beta/p')}\right) = O\left(\frac{1}{\left(\log \frac{2^\nu}{\pi}\right)^{\beta(1+\alpha p)/p}}\right);$$

and hence

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^\beta \log^T n \leqslant \log^T 2^\nu \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^\beta$$

$$= O\left(\frac{\log^T 2^\nu}{\left(\log \frac{2^\nu}{\pi}\right)^{\beta(1+ap)/p}}\right).$$

Therefore

$$\sum_{n=2}^{\infty} \rho_n^{\beta} \log^T n = \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{\beta} \log^T n$$

$$= O\left(\sum_{\nu=1}^{\infty} \frac{\log^T 2^\nu}{\left(\log \frac{2^\nu}{\pi}\right)^{\beta(1+\alpha p)/p}}\right) = O(1),$$

for $\beta > p(T+1)/(1+\alpha p)$.
This proves the first part of the theorem.

Proof of theorem 3. We shall sketch the proof. Proceeding as in the proof of Theorem D, we can obtain by using (7),

$$\left(\sum_{n=1}^{N} \left(\rho_n \left|\sin \frac{n\pi}{2N}\right|\right)^{p'}\right)^{1/p'} \leqslant \frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(x+\frac{\pi}{2N}\right) - f\left(x-\frac{\pi}{2N}\right)\right|^p dx\right)^{1/p}$$

$$= O\left(\frac{N^{-1/p}}{\left(\log \frac{N}{\pi}\right)^{(1+\alpha p)/p} \left(\log\log \frac{N}{\pi}\right)^{1+\varepsilon}}\right);$$

and hence

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{p'} = O\left(\frac{2^{-\nu p'/p}}{\left(\log \frac{2^\nu}{\pi}\right)^{(1+\alpha p)p'/p} \left(\log\log \frac{2^\nu}{\pi}\right)^{(1+\varepsilon)/p'}}\right).$$

Now, applying Hölder's inequality,

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \leqslant \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n^{p'}\right)^{1/p'} \left(\sum_{n=2^{\nu-1}+1}^{2^\nu} 1\right)^{1-1/p'}$$

$$= O\left(\frac{2^{-\nu/p}}{\left(\log \frac{2^\nu}{\pi}\right)^{(1+\alpha p)/p} \left(\log\log \frac{2^\nu}{\pi}\right)^{1+\varepsilon}} \cdot 2^{\nu/p}\right).$$

Therefore

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \log^T n \leqslant \log^T 2^\nu \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n$$

$$= O\left(\frac{\log^T 2^\nu}{\left(\log \frac{2^\nu}{\pi}\right)^{(1+\alpha p)/p} \left(\log\log \frac{2^\nu}{\pi}\right)^{1+\varepsilon}}\right)$$

$$= O\left(\frac{1}{\nu \log^{1+\varepsilon} \nu}\right), \quad \text{putting } T = \alpha + p^{-1} - 1.$$

Hence

$$\sum_{n=2}^{\infty} \rho_n \log^T n = \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \rho_n \log^T n = O\left(\sum_{\nu=1}^{\infty} \frac{1}{\nu \log^{1+\varepsilon} \nu}\right) = O(1).$$

The proof is complete.

We omit the proof of the second part of Theorem 1.

In fact, the method of proofs which we have developed above can be used to prove each and every one of the results proved above in their more general forms. Thus we have

Theorem 4. Let $0 < \alpha \leqslant 1$, $1 < p \leqslant 2$ and $h > 0$.

(i) If

$$\int_0^{2\pi} |f(x+h) - f(x)|^p \, dx =$$

$$= O\left(h (\log_1 h^{-1})^{-p} (\log_2 h^{-1})^{-p} \dots (\log_{k-1} h^{-1})^{-p} (\log_k h^{-1})^{-(1+\alpha p)}\right),$$

where $\log_1 x = \log x$ and $\log_n x = \log \log_{n-1} x$, then

$$\sum_{n=2}^{\infty} (|a_n| + |b_n|) \log^T n < \infty, \quad \text{for} \quad T < \alpha + p^{-1} - 1,$$

but not for $T = \alpha + p^{-1} - 1$ [2; Theorem 3].

(ii) If

$$\int_0^{2\pi} |f(x+h) - f(x)|^p \, dx =$$

$$= O\left(h^{\delta} (\log_1 h^{-1})^{-p} (\log_2 h^{-1})^{-p} \dots (\log_{k-1} h^{-1})^{-p} (\log_k h^{-1})^{-(1+\alpha p)}\right),$$

$$\delta = 1 + p(1-\beta)/\beta,$$

then

$$\sum_{n=2}^{\infty} (|a_n|^{\beta} + |b_n|^{\beta}) \log^T n < \infty, \quad \text{for} \quad \beta > p(T+1)/(1+\alpha p),$$

but not necessarily for $\beta = p(T+1)/(1+\alpha p)$.

(iii) If

$$\int_0^{2\pi} |f(x+h) - f(x)|^p \, dx =$$

$$= O\left(h (\log_1 h^{-1})^{-p} (\log_2 h^{-1})^{-p} \dots (\log_{k-1} h^{-1})^{-p} (\log_k h^{-1})^{-(1+\alpha p)} (\log_{k+1} h^{-1})^{-(1+\varepsilon)p}\right),$$

then

$$\sum_{n=2}^{\infty} (|a_n| + |b_n|) \log^T n < \infty,$$

for

$$T = \alpha + p^{-1} - 1.$$

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