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ON COMPLETE ELLIPTIC INTEGRALS

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1. It is well-known that the complete elliptic integrals of the first and second kind viz.,

$$K(k)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

(1.1) and

$$E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta$$

are alternative notations for hypergeometric function of Gauss:

$$K(k)=\frac{\pi}{2}F\left(\frac{1}{2},\ \frac{1}{2};\ 1;\ k^2\right)$$

(1.2) and

$$E(k)=\frac{\pi}{2}F\left(-\frac{1}{2},\ \frac{1}{2};\ 1;\ k^2\right).$$

Also the complementary integrals K' and E' have the same form as in (1.2) only instead of the module $k\in[0,1]$ there is the complementary module k' connected with k by the equation $k^2+k'^2=1$. In a previous paper [1] we have shown that

$$K(k)=\frac{\sqrt{\pi}}{2}\int_0^{\infty}e^{-x}x^{-\frac{1}{2}}L^{(0)}_{-\frac{1}{2}}(k^2x)\,dx \tag{1.3}$$

$$E(k)=\frac{\sqrt{\pi}}{2}\int_0^{\infty}e^{-x}x^{-\frac{1}{2}}L^{(0)}_{\frac{1}{2}}(k^2x)\,dx \tag{1.4}$$

where $L^{(\alpha)}_n(x)$ is the Laguerre function of degree n.

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In symbolic calculus, (1.3) and (1.4) can be written as follows

(1.5)
$$K(k)=\frac{\sqrt{\pi}}{2}L\left[x^{-\frac{1}{2}}L^{(0)}_{-\frac{1}{2}}(k^2x);\ 1\right]$$
$$=\frac{\sqrt{\pi}}{2}M\left[e^{-x}L^{(0)}_{-\frac{1}{2}}(k^2x);\ \frac{1}{2}\right]$$

(1.6)
$$E(k)=\frac{\sqrt{\pi}}{2}L\left[x^{-\frac{1}{2}}L^{(0)}_{\frac{1}{2}}(k^2x);\ 1\right]$$
$$=\frac{\sqrt{\pi}}{2}M\left[e^{-x}L^{(0)}_{\frac{1}{2}}(k^2x);\ \frac{1}{2}\right]$$

where $L[f(x);\ s]$ and $M[f(x);\ s]$ stand for the Laplace and Mellin transforms respectively of $f(x)$.

The object of this note is to add some results on the complete elliptic integrals.

2. Recently Gonźalez [2] has shown that

(2.1)
$$K\left(\sqrt{\frac{1-\lambda}{2}}\right)=\frac{\pi}{2}P_{-\frac{1}{2}}(\lambda);\quad E\left(\sqrt{\frac{1-\lambda}{2}}\right)=\frac{\pi}{4}\left[P_{-\frac{1}{2}}(\lambda)+P_{\frac{1}{2}}(\lambda)\right]$$
$$K'=\frac{\pi}{2}P_{-\frac{1}{2}}(-\lambda);\quad E'=\frac{\pi}{4}\left[P_{-\frac{1}{2}}(-\lambda)+P_{\frac{1}{2}}(-\lambda)\right]$$

where $P_n(x)$ is the Legendre function of degree n of the first kind. Here we like to point out that such representation of the complete elliptic integrals is not new. For we know [3]

$$P_{-\frac{1}{2}}(\cos\theta)=\left(\frac{1}{2}\pi\right)^{-1}K(\sin\theta/2).$$

We also know that [3, p. 173]

(2.2)
$$P_{-\frac{1}{2}}(\cosh\eta)=\left(\frac{1}{2}\pi\cosh\eta/2\right)^{-1}K(\tanh\eta/2)$$
$$Q_{-\frac{1}{2}}(\cosh\eta)=2e^{-\eta/2}K(e^{-\eta})$$
$$P_{\frac{1}{2}}(\cosh\eta)=\left(\frac{\pi}{2}\right)^{-1}e^{\eta/2}E\left[(1-e^{-2\eta})^{\frac{1}{2}}\right].$$

From (2.2) we easily derive

$$P_{-\frac{1}{2}}\left(\frac{3-\lambda}{1+\lambda}\right)=\frac{2}{\pi}\sqrt{\frac{1+\lambda}{2}}K\left(\sqrt{\frac{1-\lambda}{2}}\right)$$

$$P_{-\frac{1}{2}}\left(\frac{3+\lambda}{1-\lambda}\right)=\frac{2}{\pi}\sqrt{\frac{1-\lambda}{2}}K'$$

$$(2.3)\qquad \begin{aligned}
&Q_{-\frac{1}{2}}\left(\frac{3-\lambda}{2\sqrt{2}\sqrt{1-\lambda}}\right)=2\left(\frac{1-\lambda}{2}\right)^{\frac{1}{4}}K\left(\sqrt{\frac{1-\lambda}{2}}\right)\\
&Q_{-\frac{1}{2}}\left(\frac{3+\lambda}{2\sqrt{2}\sqrt{1+\lambda}}\right)=2\left(\frac{1+\lambda}{2}\right)^{\frac{1}{4}}K'\\
&P_{\frac{1}{2}}\left(\frac{3+\lambda}{2\sqrt{2}\sqrt{1+\lambda}}\right)=2/\pi\left(\frac{2}{1+\lambda}\right)^{\frac{1}{4}}E\left(\sqrt{\frac{1-\lambda}{2}}\right)\\
&P_{\frac{1}{2}}\left(\frac{3-\lambda}{2\sqrt{2}\sqrt{1-\lambda}}\right)=2/\pi\left(\frac{2}{1-\lambda}\right)^{\frac{1}{4}}E'.
\end{aligned}$$

Next we have from (1.3) and (1.4)

$$(2.4)\qquad K\left(\sqrt{\frac{1-\lambda}{2}}\right)=\frac{\sqrt{\pi}}{2}\int_0^\infty e^{-x}L^{(0)}_{-\frac{1}{2}}\left(\frac{1-\lambda}{2}x\right)x^{-\frac{1}{2}}\,dx$$

$$(2.5)\qquad E\left(\sqrt{\frac{1-\lambda}{2}}\right)=\frac{\sqrt{\pi}}{2}\int_0^\infty e^{-x}L^{(0)}_{\frac{1}{2}}\left(\frac{1-\lambda}{2}x\right)x^{-\frac{1}{2}}\,dx$$

$$(2.6)\qquad K'=\frac{\sqrt{\pi}}{2}\int_0^\infty e^{-x}L^{(0)}_{-\frac{1}{2}}\left(\frac{1+\lambda}{2}x\right)x^{-\frac{1}{2}}\,dx$$

$$(2.7)\qquad E'=\frac{\sqrt{\pi}}{2}\int_0^\infty e^{-x}L^{(0)}_{\frac{1}{2}}\left(\frac{1+\lambda}{2}x\right)x^{-\frac{1}{2}}\,dx.$$

We thus easily derive that

$$(2.8)\qquad \int_0^\infty e^{-x}x^{-\frac{1}{2}}L^{(0)}_{-\frac{1}{2}}\left(\frac{1-\lambda}{2}x\right)dx=\sqrt{\pi}\,P_{-\frac{1}{2}}(\lambda)$$

$$(2.9)\qquad \int_0^\infty e^{-x}x^{-\frac{1}{2}}L^{(0)}_{-\frac{1}{2}}\left(\frac{1+\lambda}{2}x\right)dx=\sqrt{\pi}\,P_{-\frac{1}{2}}(-\lambda)$$

$$(2.10)\qquad \int_0^\infty e^{-x}x^{-\frac{1}{2}}L^{(0)}_{\frac{1}{2}}\left(\frac{1-\lambda}{2}x\right)dx=\frac{\sqrt{\pi}}{2}\left[P_{-\frac{1}{2}}(\lambda)+P_{\frac{1}{2}}(\lambda)\right]$$

$$(2.11)\qquad \int_0^\infty e^{-x}x^{-\frac{1}{2}}L^{(0)}_{\frac{1}{2}}\left(\frac{1+\lambda}{2}x\right)dx=\frac{\sqrt{\pi}}{2}\left[P_{-\frac{1}{2}}(-\lambda)+P_{\frac{1}{2}}(-\lambda)\right].$$

Further we notice [4] that

$$(2.12)\qquad K(k)=\frac{\sqrt{\pi}}{2}\int_0^\infty e^{-x+\frac{1}{2}k^2x}I_0\left(\frac{1}{2}k^2x\right)x^{-\frac{1}{2}}\,dx$$

(2.13)
$$E(k) = -\frac{\sqrt{\pi}}{4}\int_0^\infty e^{-x+\frac{1}{2}k^2 x}\, I_0\left(\frac{1}{2}k^2 x\right) x^{-\frac{3}{2}}\, dx$$

where $I_0(x)$ is the modified Bessel function of the first kind of order zero.

We therefore readily obtain

(2.14)
$$K\left(\sqrt{\frac{1-\lambda}{2}}\right) = \frac{\sqrt{\pi}}{2}\int_0^\infty e^{-\frac{1}{4}(3+\lambda)x}\, I_0\left(\frac{1-\lambda}{4}x\right) x^{-\frac{1}{2}}\, dx$$

(2.15)
$$E\left(\sqrt{\frac{1-\lambda}{2}}\right) = -\frac{\sqrt{\pi}}{4}\int_0^\infty e^{-\frac{1}{4}(3+\lambda)x}\, I_0\left(\frac{1-\lambda}{4}x\right) x^{-\frac{3}{2}}\, dx$$

(2.16)
$$K' = \frac{\sqrt{\pi}}{2}\int_0^\infty e^{-\frac{1}{4}(3-\lambda)x}\, I_0\left(\frac{1+\lambda}{4}x\right) x^{-\frac{1}{2}}\, dx$$

(2.17)
$$E' = -\frac{\sqrt{\pi}}{4}\int_0^\infty e^{-\frac{1}{4}(3-\lambda)x}\, I_0\left(\frac{1+\lambda}{4}x\right) x^{-\frac{3}{2}}\, dx.$$

Comparing these results with those of Gonźalez, we derive

(2.18)
$$\int_0^\infty e^{-\frac{1}{4}(3+\lambda)x}\, I_0\left(\frac{1-\lambda}{4}x\right) x^{-\frac{1}{2}}\, dx = \sqrt{\pi}\, P_{-\frac{1}{2}}(\lambda)$$

(2.19)
$$\int_0^\infty e^{-\frac{1}{4}(3-\lambda)x}\, I_0\left(\frac{1+\lambda}{4}x\right) x^{-\frac{1}{2}}\, dx = \sqrt{\pi}\, P_{-\frac{1}{2}}(-\lambda)$$

(2.20)
$$\int_0^\infty e^{-\frac{1}{4}(3+\lambda)x}\, I_0\left(\frac{1-\lambda}{4}x\right) x^{-\frac{3}{2}}\, dx = -\sqrt{\pi}\left[P_{-\frac{1}{2}}(\lambda) + P_{\frac{1}{2}}(\lambda)\right]$$

(2.21)
$$\int_0^\infty e^{-\frac{1}{4}(3-\lambda)x}\, I_0\left(\frac{1+\lambda}{4}x\right) x^{-\frac{3}{2}}\, dx = -\sqrt{\pi}\left[P_{-\frac{1}{2}}(-\lambda) + P_{\frac{1}{2}}(-\lambda)\right].$$

3. Recently Fempl [5] has made use of Gonźalez results to derive the following connection between Legendre functions of the index $\frac{1}{2}$ and $-\frac{1}{2}$:

(3.1)
$$P_{\frac{1}{2}}(\lambda)\, P_{-\frac{1}{2}}(-\lambda) + P_{-\frac{1}{2}}(\lambda)\, P_{\frac{1}{2}}(-\lambda) = \frac{4}{\pi}.$$

We remark here that from (2.1) we easily observe

$$(3.2)\qquad \begin{aligned} P_{\frac{1}{2}}(\lambda) &= \frac{4}{\pi}\left(E-\frac{1}{2}K\right); \quad P_{-\frac{1}{2}}(\lambda)=\frac{2}{\pi}K \\ P_{\frac{1}{2}}(-\lambda) &= \frac{4}{\pi}\left(E'-\frac{1}{2}K'\right); \quad P_{-\frac{1}{2}}(-\lambda)=\frac{2}{\pi}K'. \end{aligned}$$

Putting these values we at once obtain

$$P_{\frac{1}{2}}(\lambda)\,P_{-\frac{1}{2}}(-\lambda)+P_{-\frac{1}{2}}(\lambda)\,P_{\frac{1}{2}}(-\lambda)=\frac{8}{\pi^2}\,(EK'+E'K-KK').$$

We further notice that

$$(3.3)\qquad P_{\frac{1}{2}}(\lambda)\,P_{-\frac{1}{2}}(-\lambda)-P_{-\frac{1}{2}}(\lambda)\,P_{\frac{1}{2}}(-\lambda)=\frac{8}{\pi^2}\,(EK'-E'K)$$

where the module $k=\sqrt{\frac{1-\lambda}{2}}\cdot$

Again we note from (2.4) to (2.7) that Legendre's relation

$$EK'+E'K-KK'=\pi/2$$

is the analogue of the symbolic formula

$$(3.4)\qquad \begin{aligned} &L\left[x^{-\frac{1}{2}}L^{(0)}_{\frac{1}{2}}\left(\frac{1-\lambda}{2}x\right);\ 1\right]\cdot L\left[x^{-\frac{1}{2}}L^{(0)}_{-\frac{1}{2}}\left(\frac{1+\lambda}{2}x\right);\ 1\right] \\ &+L\left[x^{-\frac{1}{2}}L^{(0)}_{\frac{1}{2}}\left(\frac{1+\lambda}{2}x\right);\ 1\right]\cdot L\left[x^{-\frac{1}{2}}L^{(0)}_{-\frac{1}{2}}\left(\frac{1-\lambda}{2}x\right);\ 1\right] \\ &-L\left[x^{-\frac{1}{2}}L^{(0)}_{-\frac{1}{2}}\left(\frac{1-\lambda}{2}x\right);\ 1\right]\cdot L\left[x^{-\frac{1}{2}}L^{(0)}_{-\frac{1}{2}}\left(\frac{1+\lambda}{2}x\right);\ 1\right]=2. \end{aligned}$$

Also from (2.3) we notice that Legendre's relation is the analogue of the following formula:

$$(3.5)\qquad \begin{aligned} &P_{\frac{1}{2}}\left(\frac{3+\lambda}{2\sqrt{2}\sqrt{1+\lambda}}\right)Q_{-\frac{1}{2}}\left(\frac{3+\lambda}{2\sqrt{2}\sqrt{1+\lambda}}\right)+P_{\frac{1}{2}}\left(\frac{3-\lambda}{2\sqrt{2}\sqrt{1-\lambda}}\right)Q_{-\frac{1}{2}}\left(\frac{3-\lambda}{2\sqrt{2}\sqrt{1-\lambda}}\right) \\ &=2+\frac{2\pi}{\sqrt{1-\lambda^2}}P_{-\frac{1}{2}}\left(\frac{3+\lambda}{1-\lambda}\right)P_{-\frac{1}{2}}\left(\frac{3-\lambda}{1+\lambda}\right). \end{aligned}$$

4. We shall now find some expansions of the complete elliptic integrals in terms of Legendre polynomials. To this end, we observe [6]

$$(4.1)\qquad P_\nu(\cos\theta)=\frac{\sin\nu\pi}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{\nu-n}-\frac{1}{\nu+n+1}\right]P_n(\cos\theta)$$

$$(\nu\neq 0,\ \pm 1,\ \pm 2,\dots;\ 0\leqslant\theta\leqslant\pi)$$

$$(4.2)\qquad Q_\nu(x)=\pi\,\frac{\cos\nu\pi\,P_\nu(x)-P_\nu(-x)}{2\sin\nu\pi}\quad \text{for } \nu\neq 0,\ \pm 1,\ \pm 2,\dots$$

Thus we derive

$$P_{-\frac{1}{2}}(\lambda)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}P_n(\lambda)$$

$$P_{\frac{1}{2}}(\lambda)=\frac{4}{\pi}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(2n+1)}{(2n-1)(2n+3)}P_n(\lambda)$$

$$Q_{-\frac{1}{2}}(\lambda)=\frac{\pi}{2}P_{-\frac{1}{2}}(-\lambda)=2\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}P_n(\lambda).$$

It follows therefore from (2.3) that

$$K\left(\sqrt{\frac{1-\lambda}{2}}\right)=2\sqrt{\frac{2}{1+\lambda}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}P_n\left(\frac{3-\lambda}{1+\lambda}\right) \tag{4.3}$$

$$K'=2\sqrt{\frac{2}{1-\lambda}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}P_n\left(\frac{3+\lambda}{1-\lambda}\right) \tag{4.4}$$

$$K\left(\sqrt{\frac{1-\lambda}{2}}\right)=\left(\frac{2}{1-\lambda}\right)^{\frac{1}{4}}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}P_n\left(\frac{3-\lambda}{2\sqrt{2}\sqrt{1-\lambda}}\right) \tag{4.5}$$

$$K'=\left(\frac{2}{1+\lambda}\right)^{\frac{1}{4}}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}P_n\left(\frac{3+\lambda}{2\sqrt{2}\sqrt{1+\lambda}}\right) \tag{4.6}$$

$$E\left(\sqrt{\frac{1-\lambda}{2}}\right)=2\left(\frac{1+\lambda}{2}\right)^{\frac{1}{4}}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)}{(2n-1)(2n+3)}P_n\left(\frac{3+\lambda}{2\sqrt{2}\sqrt{1+\lambda}}\right) \tag{4.7}$$

$$E'=2\left(\frac{1-\lambda}{2}\right)^{\frac{1}{4}}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)}{(2n-1)(2n+3)}P_n\left(\frac{3-\lambda}{2\sqrt{2}\sqrt{1-\lambda}}\right). \tag{4.8}$$

Putting $\lambda=1$, in (4.3) and (4.7) we derive in particular

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n+1)}{(2n-1)(2n+3)}.$$

Putting $\lambda=-1$, in (4.4) and (4.8) we obtain the same two summation of series.

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