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GENERALIZATION OF THE PFAFF—BILIMOVIĆ METHOD IN THE FIELD THEORY

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SUMMARY: The Pfaff—Bilimović method, applicable to mechanical problems, is extended to the field theory, by use of the method of functionals.

INTRODUCTION

Linear differential expressions called Pfaffians were first introduced to mechanics by *E. Whittaker* [1], who showed that the Hamilton equations represent Pfaff equations corresponding to a properly chosen Pfaffian. This correspondance, as well as the invariance of the Pfaff equations, resulted in the development of a method, which reduced the problem of the integration of the Hamilton equations and of canonical transformations to the problem of transformation of corresponding Pfaffians.

A. Bilimović [2, 3, 4] gave further consideration to the role of Pfaffians and Pfaff equations in mechanics, developed and generalized this method and formulated a general principle of mechanics. By giving Pfaff equations a more convenient form, he correlated them to the so-called Pfaff tensor and to the pure increments of the corresponding Pfaffians. Furthermore, he showed that by an appropriate transformation of the element of action, a Pfaffian can be obtained, which yields both Lagrange and Hamilton equations, and established thus a general principle of mechanics, named Pfaff principle, and having the character of a general phenomenological differential principle. This method was subsequently applied to a series of problems of theoretical mechanics, celestial mechanics, and geometrical optics. We are, therefore, of the opinion that the method based on the application of Pfaffians and Pfaff equations to mechanics should be named the Pfaff-Bilimović method, and the principle itself the Pfaff-Bilimović principle.

T. Andjelić [5, 6] completed this method and extended it to the mechanics of continuous media, showing that the principle can be used to obtain general differential equations of elastic bodies and viscous fluids, the element of action per unit mass being the starting point. It is thus established that the principle has the character of a universal differential principle of mechanics.

A number of works is dedicated to the application of this method to mechanics of the system of particles. Thus, *P. Musen* [7] applied it to the perturbation of vector elements, and *V. Vujičić* [8] to the problems of motion with variable mass.

In a previous work [9] we showed that the Pfaff—Bilimović method can be generalized to the case of more independent variables, so that it is applicable to the theoretical physics as well, namely to all those branches in which quantities analogous to kinetic and potential energy can be found. Using the notion of the functional derivative, and introducing the notion of the functional differential, we generalized the Pfaffians, Pfaf equations, and the very Pfaff—Bilimović principle. Thereafter, we applied the method to the theory of elasticity, thermodynamics, electrodynamics, and quantum mechanics, to obtain the differential equations of the corresponding phenomena.

The deeper meaning of this generalization is to be found in the theory of functionals, created by *V. Volterra* [10], which is applicable to the theoretical physics [11]. In this theory, the notion of the functional itself represents a generalization of the notion of the function, when the number of independent variables tends to infinity, and in the functional calculus the role of primary notion is played by the functional derivative and functional differential, which are generalizations of the partial derivative and total differential.

In this work we attempt to generalize the Pfaff-Bilimović method of mechanics to the field theory, on the ground of the theory of functionals. In the first part we purport to introduce convenient definitions of functional derivative and functional differential, applicable to the functionals given in the form of integrals, and to give the corresponding analytical expressions. In the second part we proceed to generalize the Pfaffians on account of the introduced notions of the theory of functionals, to expose the generalized Pfaff-Bilimović method, and the very Pfaff-Bilimović principle.

1. DERIVATIVES AND DIFFERENTIALS OF A FUNCTIONAL

Notion of a Functional. — Let φ_i be certain functions of coordinates x_j and a parameter τ in a k-dimensional Euclidean space:

$$\varphi_i = \varphi_i(x_j, \tau) \qquad \begin{pmatrix} i = 1, \ 2, \ldots, n \\ j = 1, \ 2, \ldots, k \end{pmatrix} \tag{1.1}$$

and let us define:

$$\varphi_{ij} = \frac{\partial \varphi_i}{\partial x_j}. \tag{1.2}$$

We shall now consider a compound function of the form

$$\mathfrak{F} = \mathfrak{F}(\varphi_i, \varphi_{ij}, x_j) \tag{1.3}$$

and its volume integral over the entire region of definition of the functions φ_i:

$$F = \int_V \mathfrak{F}\, dV; \qquad dV = \prod dx_i. \tag{1.4}$$

The value of this integral clearly depends on both form of all functions $\varphi_i(x_j, \tau)$, and the value of the parameter τ.

Let us now introduce the notion of a functional, playing the basic role in our further considerations. As is well known, a quantity f depending on the independent variable x in a determined manner, so that one or more values of f correspond to each value of x; is called a *function*, and denoted by $f(x)$. This function determines the mapping of a set x into the set $f(x)$. If, however, a quantity F depends on a function $f(x)$ as independent variable, in such a way that one or more numerical values of F correspond to every form of the function $f(x)$, then F is a *functional* of the function $f(x)$. To point out the difference from the ordinary function, this type of dependence is usually denoted by square brackets:

$$F = F[f(x)] \tag{1.5}$$

and this functional coordinates a number F to the set of all values of $f(x)$, i. e. determines a mapping of the set $f(x)$ into a number F. It is in this sense that the functional may be thought of as being dependent on an infinite number of variables, namely on all values of the function $f(x)$.

According to this definition of a functional, the integral (1.4) is clearly seen to represent a functional of all φ_i, being dependent on the forms of all these functions, so that one value of this integral corresponds to every set of these, i. e.

$$F = F[\varphi_i(x_j, \tau)], \tag{1.6}$$

φ_i denoting here the aggregate of all corresponding functions.

On the other hand, this integral depends also on the parameter τ, so that one value of the integral corresponds to each value of τ. The integral is, therefore, at the same time an ordinary function of τ:

$$F = F(\tau). \tag{1.7}$$

Functional Derivative. — Each point of the Euclidean space with coordinates $x_j (j = 1, 2, \ldots, k)$ of the region V where the functions $\varphi_i(x_j, \tau)$ are defined, associated with any fixed value of the parameter τ, has a corresponding set of values of $\varphi_i(x_j, \tau)$. We shall supposse that these functions have the property of being τ — independent at the boundary of V, i. e.

$$\left(\frac{\partial \varphi_i}{\partial \tau}\right)_s = 0. \tag{1.8}$$

Let us now consider a point M and a small volume ΔV surrounding it; let the i-th function acquire an arbitrary increment $\Delta \varphi_i$ at the points within ΔV, and let its form remain unaltered without this volume. We have thus obtained another function

$$\varphi_i' = \varphi_i + \Delta \varphi_i \tag{1.9}$$

where:

$$\Delta \varphi_i \begin{cases} \neq 0 & \text{within } \Delta V \\ = 0 & \text{at the boundary and without } \Delta V, \end{cases} \tag{1.10}$$

$\Delta \varphi_i$ evidently having the character of the variation of the function considered.

Let $\overline{\Delta \varphi_i}$ be the mean value of these increments in ΔV. We shall henceforth use the symbol $[\ldots, \varphi_i, \ldots]$ whenever we refer to the i-th member of the set of functions φ_i only, i. e. φ_i in this case does not represent the entire set of the functions, but its i-th member only. Then, the *functional derivative of the functional F with respect to the function* φ_i at the point M, sometimes also called the *variational derivative of the functional,* will be defined by the following formule:

$$\frac{\delta F}{\delta \varphi_i} = \lim_{\overline{\Delta \varphi_i}, \Delta V \to 0} \frac{F[\ldots, \varphi_i + \Delta \varphi_i, \ldots] - F[\ldots, \varphi_i, \ldots]}{\overline{\Delta \varphi_i} \cdot \Delta V}. \tag{1.11}$$

The expression in the denumerator, according to (1.3) and (1.4) has the explicite form:

$$\begin{aligned} &F[\ldots, \varphi_i + \Delta\varphi_i, \ldots] - F[\ldots, \varphi_i, \ldots] = \\ &= \int_V \mathfrak{F}(\ldots, \varphi_i + \Delta\varphi_i, \varphi_{ij} + \Delta\varphi_{ij}, \ldots)\, dV - \int_V \mathfrak{F}(\ldots, \varphi_i, \varphi_{ij}, \ldots)\, dV. \end{aligned} \tag{1.12}$$

Expanding the integrand into the Taylor series, bearing in mind the conditions (1.10) and neglecting the higher order terms, one obtains:

$$F[\ldots,\ \varphi_i+\Delta\varphi_i,\ \ldots]-F[\ldots,\ \varphi_i,\ \ldots]=$$

$$=\int\limits_{\Delta V}\{\mathfrak{F}(\ldots\ \varphi_i+\Delta\varphi_i,\ \varphi_{ij}+\Delta\varphi_{ij},\ \ldots)-\mathfrak{F}(\ldots,\ \varphi_i,\ \varphi_{ij},\ \ldots)\}\,dV=$$

$$=\int\limits_{\Delta V}\left(\Delta\varphi_i\frac{\partial\mathfrak{F}}{\partial\varphi_i}+\sum_{j=i}^{k}\Delta\varphi_{ij}\frac{\partial\mathfrak{F}}{\partial\varphi_{ij}}\right)dV.$$

Each integral under the summation sign can be transformed by partial integration, with the conditions (1.10) still in mind, the order of operations $\frac{\partial}{\partial x_j}$ and Δ being interchangeable. It should be noted that the symbol $\frac{\partial}{\partial x_j}$ represents in fact the operation of total partial derivation with respect to x_j; i. e. differentiation over all functions depending on x_j: we shall henceforth use the symbol $\frac{d}{dx_j}$ for this operation. It follows:

$$\int\limits_{\Delta V}\frac{\partial\mathfrak{F}}{\partial\varphi_{ij}}\Delta\varphi_{ij}\,dV=\int\limits_{\Delta V}\frac{\partial\mathfrak{F}}{\partial\varphi_{ij}}\frac{d}{dx_j}\Delta\varphi_i\,dV=$$

$$=\underbrace{\int\ldots\int}_{(k-1)}dx_1\ldots dx_{j-1}\,dx_{j+1}\ldots dx_k\cdot\int\frac{\partial\mathfrak{F}}{d\varphi_{ij}}\,d_j\Delta\varphi_i=$$

$$=\underbrace{\int\ldots\int}_{(k-1)}dx_1\ldots dx_{j-1}\,dx_{j+1}\ldots dx_k\left\{\left|\frac{\partial\mathfrak{F}}{\partial\varphi_{ij}}\Delta\varphi_i\right|_{(x_j)_1}^{(x_j)_2}-\right.$$

$$\left.-\int\Delta\varphi_i\,d_j\frac{\partial\mathfrak{F}}{\partial\varphi_{ij}}\right\}=-\int\limits_{\Delta V}\frac{d}{dx_j}\frac{\partial\mathfrak{F}}{\partial\varphi_{ij}}\Delta\varphi_i\,dV$$

The theorem of the mean value now yields:

$$F[\ldots,\ \varphi_i+\Delta\varphi_i,\ldots]-F[\ldots,\ \varphi_i,\ \ldots]=\int\limits_{\Delta V}\left(\frac{\partial\mathfrak{F}}{\partial\varphi_i}-\sum_{j=1}^{k}\frac{d}{dx_j}\frac{\partial\mathfrak{F}}{\partial\varphi_{ij}}\right)\Delta\varphi_i\,dV=$$

(1.13)
$$=\overline{\left(\frac{\partial\mathfrak{F}}{\partial\varphi_i}-\sum_{j=1}^{k}\frac{d}{dx_j}\frac{\partial\mathfrak{F}}{\partial\varphi_{ij}}\right)}\,\overline{\Delta\varphi_i}\,\Delta V.$$

Under the assumption that the function $\mathfrak{F}$ and its first and second order derivatives are continuous at M, (1.11) becomes:

$$\frac{\delta F}{\delta\varphi_i}=\lim_{\overline{\Delta\varphi_i},\,\Delta V\to 0}\frac{\overline{\left(\frac{\partial\mathfrak{F}}{\partial\varphi_i}-\sum_{j=1}^{k}\frac{d}{dx_j}\frac{\partial\mathfrak{F}}{\partial\varphi_{ij}}\right)}\,\overline{\Delta\varphi_i}\,\Delta V}{\overline{\Delta\varphi_i}\,\Delta V}=\left(\frac{\partial\mathfrak{F}}{\partial\varphi_i}-\sum_{j=1}^{k}\frac{d}{dx_j}\frac{\partial\mathfrak{F}}{\partial\varphi_{ij}}\right)_M$$

or, shorter:

(1.14)
$$\frac{\delta F}{\delta\varphi_i}=\frac{\partial\mathfrak{F}}{\partial\varphi_i}-\sum_{j=1}^{k}\frac{d}{dx_j}\frac{\partial\mathfrak{F}}{\partial\varphi_{ij}}.$$

This is the well-known analytical expression for the functional derivative; it is easily seen that the functional derivative in a point, as a function of position, depends on coordinates through functions $\varphi_i(x_j, \tau)$, being thus dependent both on their forms and on the coordinates of the point considered. The functional derivative is, therefore, a functional of all functions φ_i, but not of the form (1.4), and an ordinary function of the coordinates x_j:

$$\frac{\delta F}{\delta \varphi_i} = \frac{\delta F}{\delta \varphi_i}[\varphi_i], \qquad \frac{\delta F}{\delta \varphi_i} = \frac{\delta F}{\delta \varphi_i}(x_j) \tag{1.15}$$

Equation (1.14) shows directly that for the sum of two functionals and for the product of a functional and a constant the following relations hold:

$$\frac{\delta}{\delta \varphi_i}(F_1 + F_2) = \frac{\delta F_1}{\delta \varphi_i} + \frac{\delta F_2}{\delta \varphi_i}; \qquad \frac{\delta}{\delta \varphi_i}(CF) = C\frac{\delta F}{\delta \varphi_i} \tag{1.16}$$

the operator $\frac{\delta}{\delta \varphi_i}$ of functional differentiation being thus shown to be linear.

There is, however, another type of derivative of the functional (1.4), based on the fact that it is also an ordinary function of the parameter τ. We can, therefore, introduce the *derivative of a functional with respect to a parameter*, $\frac{dF}{d\tau}$.

The derivative of the integral with respect to a parameter being equal to the integral of the derivative with respect to it, we have:

$$\frac{dF}{d\tau} = \frac{d}{d\tau}\int_V \mathfrak{F}\, dV = \int_V \frac{d\mathfrak{F}}{d\tau}\, dV$$

or:

$$\frac{dF}{d\tau} = \int_V \left(\sum_{i=1}^{n} \frac{\partial \mathfrak{F}}{\partial \varphi_i}\frac{\partial \varphi_i}{\partial \tau} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial \mathfrak{F}}{\partial \varphi_{ij}}\frac{\partial \varphi_{ij}}{\partial \tau} \right) dV. \tag{1.17}$$

Integral under the double summation sign can be transformed by partial integration, interchanging the order of operations $\frac{\partial}{\partial x_j}$ and $\frac{\partial}{\partial \tau}$, bearing in mind that $\frac{\partial}{\partial x_j}$ represents the total partial differentiation with respect to x_j:

$$\int_V \frac{\partial \mathfrak{F}}{\partial \varphi_{ij}}\frac{\partial \varphi_{ij}}{\partial \tau}\, dV = \int_V \frac{\partial \mathfrak{F}}{\partial \varphi_{ij}}\frac{d}{dx_j}\frac{\partial \varphi_i}{\partial \tau}\, dV =$$

$$= \underbrace{\int \cdots \int}_{(k-1)} dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_k \int \frac{\partial \mathfrak{F}}{\partial \varphi_{ij}}\, d_j \frac{\partial \varphi_i}{\partial \tau} =$$

$$= \underbrace{\int \cdots \int}_{(k-1)} dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_k \left\{ \left| \frac{\partial \mathfrak{F}}{\partial \varphi_{ij}}\frac{\partial \varphi_i}{\partial \tau} \right|_{(x_j)_1}^{(x_j)_2} - \int \frac{\partial \varphi_i}{\partial \tau}\, d_j \frac{\partial \mathfrak{F}}{\partial \varphi_{ij}} \right\}$$

The integrated part cancels out in view of the conditions (1.8), so that we have:

$$\int_V \frac{\partial \mathfrak{F}}{\partial \varphi_{ij}} \frac{\partial \varphi_{ij}}{\partial \tau} dV = -\int_V \frac{d}{dx_j} \frac{\partial \mathfrak{F}}{\partial \varphi_{ij}} \frac{\partial \varphi_i}{\partial \tau} dV$$

and the relation (1.17) becomes

$$\frac{dF}{d\tau} = \int_V \sum_{i=1}^{n} \left(\frac{\partial \mathfrak{F}}{\partial \varphi_i} - \sum_{j=1}^{k} \frac{d}{dx_j} \frac{\partial \mathfrak{F}}{\partial \varphi_{ij}} \right) \frac{\partial \varphi_i}{\partial \tau} dV$$

and, finally, in view of (1.14):

$$\frac{dF}{d\tau} = \int_V \sum_{i=1}^{n} \frac{\delta F}{\delta \varphi_i} \frac{\partial \varphi_i}{\partial \tau} dV. \tag{1.18}$$

This equation determines the derivative of the functional of the form (1.4) with respect to the parameter τ, and is easily seen to represent a generalization of the formula for derivatives of compound functions.

Functional Differential. — Let us divide the entire region V of definition of functions $\varphi_i(x_j, \tau)$ into a very large number N of very small cells ΔV_l, each containing a point M_l, in such a manner that the volume of each cell tends to zero when the number of cells tends to infinity. Now, let us alter the form of all functions φ_i in each cell, giving these functions arbitrary increments satisfying (1.10). Denoting by $\{\ \ \}_l$ the value of any quantity in the l-th cell, the *functional differential of the functional F with respect to* functions φ_i, also called *variation of the functional*, will be defined by:

$$\delta F = \lim_{N \to \infty} \sum_{l=1}^{N} \{F[\varphi_i + \Delta\varphi_i] - F[\varphi_i]\}_l. \tag{1.19}$$

As the increments $\Delta\varphi_i$ have the character of variations, the same will hold for functional differentials, which is, therefore, denoted by the symbol δ.

The expression in brackets can be transformed by addition and subtraction of suitable expressions:

$$F[\varphi_i + \Delta\varphi_i] - F[\varphi_i] = F[\varphi_1 + \Delta\varphi_1, \varphi_2 + \Delta\varphi_2, \ldots, \varphi_n + \Delta\varphi_n] -$$

$$-F[\varphi_1, \varphi_2 + \Delta\varphi_2, \ldots, \varphi_n + \Delta\varphi_n] + F[\varphi_1, \varphi_2 + \Delta\varphi_2, \varphi_3 + \Delta\varphi_3, \ldots, \varphi_n + \Delta\varphi_n] -$$

$$-F[\varphi_1, \varphi_2, \varphi_3 + \Delta\varphi_3, \ldots, \varphi_n + \Delta\varphi_n] + \ldots + F[\varphi_1, \varphi_2, \ldots, \varphi_n + \Delta\varphi_n] -$$

$$-F[\varphi_1, \varphi_2, \ldots, \varphi_n];$$

in other words:

$$F[\varphi_i + \Delta\varphi_i] - F[\varphi_i] = \sum_{i=1}^{n} \{F[\ldots, \varphi_i + \Delta\varphi_i, \ldots] - F[\ldots, \varphi_i, \ldots]\}$$

each of the summation terms being determined by (1.13), it follows:

$$\{F[\varphi_i + \Delta\varphi_i] - F[\varphi_i]\}_l = \left\{ \sum_{i=1}^{n} \left(\frac{\partial \mathfrak{F}}{\partial \varphi_i} - \sum_{j=1}^{k} \frac{d}{dx_j} \frac{\partial \mathfrak{F}}{\partial \varphi_{ij}} \right) \overline{\Delta\varphi_i} \right\}_l \Delta V_l. \tag{1.20}$$

As ΔV_l tends to zero, the mean increment $(\overline{\Delta\varphi_i})_l$ will tend to the increment of the corresponding function at the point M_l, the value of this increment being dependent on the manner the functions φ_i are given arbitrary increments $\Delta\varphi_i$ in this cell. Let us designate this limiting value by

$$\delta\varphi_i = (\delta\varphi_i)_{M_l} = \lim_{\Delta V_l \to 0} (\overline{\Delta\varphi_i})_l \tag{1.21}$$

emphasising thus that it has the character of variation of a function, too. This guantity is a function of coordinates of M and of τ, and as a rule is a first order infinitesimal.

Equation (1.19) combined with (1.20) then yields:

$$\delta F = \lim_{N\to\infty} \sum_{l=1}^{N} \left\{ \sum_{i=1}^{n} \overline{\left(\frac{\partial \mathfrak{F}}{\partial \varphi_i} - \sum_{j=1}^{k} \frac{d}{dx_j} \frac{\partial \mathfrak{F}}{\partial \varphi_{ij}} \right) \overline{\Delta \varphi_i}} \right\}_l \Delta V_l$$

and, when the number of the cells tends to infinity in the manner stated above, the summation over index l becomes definite integral over volume V, while the expression in small brackets, in view of (1.14) tends to $\frac{\delta F}{\delta\varphi_i}$, and $(\overline{\Delta\varphi_i})_l$ to $\delta\varphi_i$, according to (1.21), so that it follows:

$$\delta F = \int_V \sum_{i=1}^{n} \frac{\delta F}{\delta\varphi_i} \delta\varphi_i \, dV. \tag{1.22}$$

This is the analytical form of the expression for functional differential. It is to be noted that functional differential depends on the forms of both φ_i and $\delta\varphi_i$; it is, therefore, a functional of all φ_i and their variations $\delta\varphi_i$ of the form (1.4):

$$\delta F = \delta F [\varphi_i, \delta\varphi_i] \tag{1.23}$$

being also linear with respect to $\delta\varphi_i$.

As in the case of functional derivatives, we can introduce here another type of a differential, too, based on the fact that (1.4) is an ordinary function of a parameter τ. It is possible, therefore, to define the *differential of a functional with respect to a parameter* τ, *dF*.

According to (1.18) we have:

$$dF = \frac{dF}{d\tau} d\tau = \int_V \sum_{i=1}^{n} \frac{\delta F}{\delta\varphi_i} \frac{\partial\varphi_i}{\partial\tau} d\tau \, dV \tag{1.24}$$

If we now consider:

$$d\varphi_i = \sum_{j=1}^{k} \frac{\partial\varphi_i}{\partial x_j} dx_j + \frac{\partial\varphi_i}{\partial\tau} d\tau$$

and make use of the mutual independance of x_j and τ:

$$\frac{d\varphi_i}{d\tau} = \sum_{j=1}^{k} \frac{\partial\varphi_i}{\partial x_j} \frac{dx_j}{d\tau} + \frac{\partial\varphi_i}{\partial\tau} = \frac{\partial\varphi_i}{\partial\tau}$$

we obtain:

$$\frac{\partial \varphi_i}{\partial \tau} d\tau = \frac{d \varphi_i}{d\tau} d\tau = d\varphi_i$$

where it should be borne in mind that the quantities $d\varphi_i$, as well as $\delta\varphi_i$, are functions of coordinates x_j and of the parameter τ; expression (1.24) yields then:

$$dF = \int_V \sum_{i=1}^{n} \frac{\delta F}{\delta \varphi_i} d\varphi_i \, dV. \tag{1.25}$$

This equality determines the differential of the functional (1.4) with respect to the parameter τ, representing thus a generalization of the formula for total differential of a function.

If $k=0$, i. e. if $\varphi_i(x_j, \tau)$ depend only on the parameter τ, (1.4) is reduced to the zero-fold integral, viz. to the integrand $\mathfrak{F}$ itself, and in this case the functional considered is reduced to an ordinary function. The functional derivative (1.14) is then similarly reduced to the ordinary partial derivative, and the functional differential (1.22) as a zero-fold integral, to the ordinary total differential. This fact clearly shows how the notion of a functional really represents a generalization of the notion of a function, and, likewise, notions of the functional derivative and the functional differential are generalizations of those of the partial derivative and the total differential.

2. GENERALIZED PFAFF — BILIMOVIĆ METHOD

Functional Pfaffians. — Consider a set of functions of the form (1.3):

$$\mathfrak{F}_i = \mathfrak{F}_i(\varphi_i, \varphi_{ij}, x_j) \qquad (i = 1, 2, \ldots, n) \tag{2.1}$$

and use them to construct a linear differential form:

$$d\Phi = \sum_{i=1}^{n} \mathfrak{F}_i \, d\varphi_i \tag{2.2}$$

where $d\varphi_i$ denotes differentials of φ_i. The volume integral of this form is:

$$\Phi = \int \sum_{i=1}^{n} \mathfrak{F}_i \, d\varphi_i \, dV \tag{2.3}$$

and it will be called *functional Pfaffian.* We shall always tacitly assume that the integral is to be extended over the whole region of definition of φ_i.

An expression thus defined represents a functional of all functions φ and all differentials $d\varphi_i$:

$$\Phi = \Phi[\varphi_i, d\varphi_i] \tag{2.4}$$

and it plays the basic role in what follows.

Functional Pfaff Equations. — According to (1.14), we have for the functional Pfaffian (2.3):

$$\frac{\delta \Phi}{\delta \varphi_i} = \frac{\partial}{\partial \varphi_i}\left(\sum_{l=1}^{n} \mathfrak{F}_l \, d\varphi_l\right) - \sum_{j=1}^{k} \frac{d}{dx_j} \frac{\partial}{\partial \varphi_{ij}} \left(\sum_{l=1}^{n} \mathfrak{F}_l \, d\varphi_l\right) =$$

$$= \sum_{l=1}^{n} \left(\frac{\partial \mathfrak{F}_l}{\partial \varphi_i} - \sum_{j=1}^{k} \frac{d}{dx_j} \frac{\partial \mathfrak{F}_l}{\partial \varphi_{ij}}\right) d\varphi_l .$$

Introducing now the quantities F_i:

$$F_i = \int \mathfrak{F}_i \, dV \tag{2.5}$$

the above equality yields:

$$\frac{\delta \Phi}{\delta \varphi_i} = \sum_{l=1}^{n} \frac{\delta F_l}{\delta \varphi} \, d\varphi_l . \tag{2.6}$$

Multiplying both sides by $d\varphi_i$, summing up with respect to the index i, and then integrating over V, we obtain:

$$\int \sum_{i=1}^{n} \frac{\delta \Phi}{\delta \varphi_i} \, d\varphi_i \, dV = \int \sum_{i=1}^{n} \sum_{l=1}^{n} \frac{\delta F_l}{\delta \varphi_i} \, d\varphi_l \, d\varphi_i \, dV. \tag{2.7}$$

On the other hand, in view of (1.25) we have:

$$dF_i = \int \sum_{l=1}^{n} \frac{\delta F_i}{\delta \varphi_l} \, d\varphi_l \, dV$$

furthermore, since the operations of differentiting with respect to a parameter and integrating over V are interchangeable, we have:

$$dF_i = d \int \mathfrak{F}_i \, dV = \int d\mathfrak{F}_i \, dV$$

so that the last two equalities yield:

$$\int d\mathfrak{F}_i \, dV = \int \sum_{l=1}^{n} \frac{\delta F_i}{\delta \varphi_l} \, d\varphi_l \, dV \tag{2.8}$$

Multiplying next both sides by the mean value $\overline{d\varphi_i}$ of the differential $d\varphi_i$ in the region V, and summing up with respect to i, one obtains:

$$\sum_{i=1}^{n} \overline{d\varphi_i} \int d\mathfrak{F}_i \, dV = \sum_{i=1}^{n} \overline{d\varphi_i} \int \sum_{l=1}^{n} \frac{\delta F_i}{\delta \varphi_l} \, d\varphi_l \, dV$$

or, applying the theorem of the mean value:

$$\int \sum_{i=1}^{n} d\mathfrak{F}_i \, d\varphi_i \, dV = \int \sum_{i=1}^{n} \sum_{l=1}^{n} \frac{\delta F_i}{\delta \varphi_l} \, d\varphi_l \, d\varphi_i \, dV, \tag{2.9}$$

Subtracting (2.7) from (2.9), it results:

$$\int \sum_{i=1}^{n} \left(d\mathfrak{F}_i - \frac{\delta \Phi}{\delta \varphi_i} \right) d\varphi_i \, dV = \int \sum_{i=1}^{n} \sum_{l=1}^{n} \left(\frac{\delta F_i}{\delta \varphi_l} - \frac{\delta F_l}{\delta \varphi_i} \right) d\varphi_l \, d\varphi_i \, dV. \tag{2.10}$$

If in the second double sum on the right first the indices i and l, and then the order of summation are interchanged, the following will result:

$$\sum_{i=1}^{n} \sum_{l=1}^{n} \frac{\delta F_l}{\delta \varphi_i} \, d\varphi_l \, d\varphi_i = \sum_{l=1}^{n} \sum_{i=1}^{n} \frac{\delta F_i}{\delta \varphi_l} \, d\varphi_i \, d\varphi_l = \sum_{i=1}^{n} \sum_{l=1}^{n} \frac{\delta F_i}{\delta \varphi_l} \, d\varphi_l \, d\varphi_i$$

which shows clearly the right-hand side of (2.10) to be equal to zero; thus:

$$\int \sum_{i=1}^{n} \left(d\mathfrak{F}_i - \frac{\delta \Phi}{\delta \varphi_i} \right) d\varphi_i \, dV = 0. \tag{2.11}$$

For this equality to hold, it is sufficient that

$$d\mathfrak{F}_i = \frac{\delta \Phi}{\delta \varphi_i} \qquad (i = 1, 2, \dots, n). \tag{2.12}$$

It is to be noted that (2.12) is not necessary for (2.11), i. e. if (2.12) is valid, (2.11) will hold, but this is not the unique possibility. Conditions (2.12) are, therefore, not to be regarded as consequences of our former considerations; they merely represent a system of n equations in functional form, attached to the functional Pfaffian (2.3). The equations (2.12) will be called *functional Pfaff equations;* they are easily derivable for any function φ_i, if only the functional Pfaffian is given in the form (2.3).

If $k=0$, i. e. if functions $\varphi_i(x_j, \tau)$ depend on τ only, functionals F_i become ordinary functions $\mathfrak{F}_i$, functional derivatives and differentials reduce to ordinary partial derivatives and total differentials, so that functional Pfaffians (2.3) and equations (2.12) yield ordinary Pfaffians and Pfaff equations.

Properties of functional Pfaff Equations. — Functional Pfaff equations have properties similar to those possessed by ordinary Pfaff equations. Their principal properties refer to the equivalence; two functional Pfaffians are considered to be equivalent if they give rise to equivalent systems of functional Pfaff equations; this will be denoted by the usual equivalence symbol$\sim$.

Let us first consider two functional Pfaffians, Φ and $k\Phi$, k being a constant. According to (2.3) we have:

$$k\Phi = \int \sum_{i=1}^{n} k\,\mathfrak{F}_i\, d\varphi_i\, dV \tag{2.13}$$

so that the corresponding functional Pfaff equations are:

$$d(k\,\mathfrak{F}_i) = \frac{\delta (k\Phi)}{\delta \varphi_i}. \tag{2.14}$$

Upon division by k, these equations coincide with those for Φ, so that

$$k\Phi \sim \Phi \tag{2.15}$$

i.e. functional Pfaffians differing in a multiplicative constant are equivalent.

Next, consider functional Pfaffian $\Phi + dG$, where G is an arbitrary functional of the variables considered:

$$G = \int \mathfrak{G}\, dV = G[\varphi_i]; \tag{2.16}$$

in view of (1.25) we have:

$$\Phi + dG = \int \sum_{i=1}^{n} \mathfrak{F}_i\, d\varphi_i\, dV + \int \sum_{i=1}^{n} \frac{\delta G}{\delta \varphi_i}\, d\varphi_i\, dV$$

i. e.

$$\Phi + dG = \int \sum_{i=1}^{n} \left(\mathfrak{F}_i + \frac{\delta G}{\delta \varphi_i} \right) d\varphi_i\, dV, \tag{2.17}$$

so that the corresponding functional Pfaff equations are:

$$d\left(\mathfrak{F}_i + \frac{\delta G}{\delta \varphi_i} \right) = \frac{\delta (\Phi + dG)}{\delta \varphi_i}. \tag{2.18}$$

Since the operations of functional derivation with respect to a function, and differentiation of the functional with respect to a parameter are independent and thus interchangeable, we have:

$$d\frac{\delta G}{\delta \varphi_i} = \frac{\delta (dG)}{\delta \varphi_i} \tag{2.19}$$

so that the second terms on both sides of (2.18) cancel out, and it remains:

$$d\mathfrak{F}_i = \frac{\delta \Phi}{\delta \varphi_i}$$

which shows that two functional Pfaffians differing by the differential of an arbitrary functional with respect to a parameter are equivalent:

$$\Phi + dG \sim \Phi. \tag{2.20}$$

Finally, if a substitution of functions φ_i is performed in Φ, a transformed Pfaffian Φ^* is obtained, but the very way of forming functional Pfaff equations makes it evident that the systems corresponding to Φ^* and Φ are equivalent. This property makes it possible to perform substitutions in the equations by the mere transformation of the corresponding Pfaffian.

Element of Action as a Functional Pfaffian. — Consider now a physical field, and assume that it can be described by a Lagrangian L. The action is then defined by:

$$W = \int_{t_0}^{t_1} L\, dt, \tag{2.21}$$

where the Lagrangian can be given an integral representation:

$$L = \int \mathfrak{L}\, dV; \quad dV = dx_1\, dx_2\, dx_3. \tag{2.22}$$

Let the field considered be determined by certain *functions of field.*

$$\psi_i = \psi_i(x_j, t) \qquad \begin{pmatrix} i = 1, 2, \dots n \\ j = 1, 2, 3. \end{pmatrix}. \tag{2.23}$$

(In the case of electromagnetic field, for instance, the functions of field are scalar potential, and the three components of vector potential.). Then, $\mathfrak{L}$ is a certain function of the functions of field and their derivatives as well as of the independent variables:

$$\mathfrak{L} = \mathfrak{L}\,(\psi_i, \psi_{ij}, \psi_{it}, x_j, t) \tag{2.24}$$

the index t here denots the time derivatives.

The Lagrangian (2.22) is now seen to be of the form (1.4), i. e. it represents a functional. Equations (1.3) and (1.6) show that for a functional depending on certain functions φ_i, the corresponding integrand is a function of both φ_i and their coordinate derivatives φ_{ij}, as well as of the coordinates x_j themselves. In (2.24), however, the time derivatives ψ_{it} are also present, along with the time t, and all have the form (1.1). In this case, therefore, not only ψ_i are to be regarded as φ_j, but also ψ_{it} and t, i. e.:

(2.25)
$$\varphi_j = \begin{cases} \psi_i & \text{for } j = i \\ \psi_{it} & \text{for } j = n + i \\ t & \text{for } j = 2n + 1 \end{cases}$$

and the Lagrangian is a functional of the following variables:

(2.26)
$$L = L\,[\psi_i,\ \psi_{it},\ t].$$

It is clearly seen that the arguments of Lagrangian, conceived as a functional in the field theory, are the same as those of Lagrangian in the mechanics of the particle-system. This coincidence brings about a characteristic analogy between the equations in the mechanics of the system, and those in the field theory.

Consider now an element of action (2.21), which is a functional Pfaffian:

(2.27)
$$\Phi = L dt$$

In view of (2.22) and of the fact that dt is independent of the coordinates, this expression can be written in the form:

(2.28)
$$\Phi = \int \mathfrak{L}\, dt\, dV$$

and it is this form of the element of action, that will serve us as a starting point in our proceeding to obtain the field differential equations.

Lagrange Equations. — Let us now transform (2.28) by adding and subtracting $\sum\limits_{i=1}^{n} \frac{\delta L}{\delta \psi_{it}} d\psi_i$ under the integration sign; it is thus possible to have it in the form:

(2.29)
$$\Phi = \int \left\{ \sum_{i=1}^{n} \frac{\delta L}{\delta \psi_{it}} d\psi_i + \sum_{i=1}^{n} 0 \cdot d\psi_{it} - \left(\sum_{i=1}^{n} \frac{\delta L}{\delta \psi_{it}} \psi_{it} - \mathfrak{L} \right) dt \right\} dV.$$

A comparison with (2.3) will show that in the Pfaffian considered, the role of functions φ is played by ψ_i, ψ_{it} and t, in accordance with (2.26).

Functional Pfaff equations (2.12) corresponding to the variables ψ_i are then:

(2.30)
$$d \frac{\delta L}{\delta \psi_{it}} = \frac{\delta \Phi}{\delta \psi_i} \qquad (i = 1, 2, \ldots, n).$$

Since:

$$\frac{\delta \Phi}{\delta \psi_i} = \frac{\delta}{\delta \psi_i} (L\, dt) = \frac{\delta L}{\delta \psi_i} dt$$

we have

$$d \frac{\delta L}{\delta \psi_{it}} = \frac{\delta L}{\delta \psi_i} dt$$

or, finally,

$$\frac{d}{dt}\frac{\delta L}{\delta \psi_{it}}-\frac{\delta L}{\delta \psi_i}=0 \qquad (i=1, 2, \ldots, n). \tag{2.31}$$

These are the *Lagrange equations* of the field theory, i. e. the differential equations of the field. Other functional Pfaff equations, those for variables ψ_{it} and t, yield identities only, as is easily seen on the ground of Lagrange equations.

Hamilton Equations. — We shall proceed now by introducing *momentum densities*, according to the formula:

$$\pi_i=\frac{\delta L}{\delta \psi_{it}} \qquad (i=1, 2, \ldots, n). \tag{2.32}$$

Since the Lagrangian density $\mathfrak{L}$ is independent of second order derivatives, it follows, according to (1.14):

$$\frac{\delta L}{\delta \psi_{it}}=\frac{\partial \mathfrak{L}}{\partial \psi_{it}}-\sum_{j=1}^{3}\frac{d}{d x_j}\frac{\partial \mathfrak{L}}{\partial \psi_{itj}}=\frac{\partial \mathfrak{L}}{\partial \psi_{it}} \qquad \left(\psi_{itj}=\frac{\partial^2 \psi_i}{\partial t\,\partial x_j}\right),$$

so that we have:

$$\pi_i=\frac{\partial \mathfrak{L}}{\partial \psi_{it}} \qquad (i=1, 2, \ldots, n). \tag{2.33}$$

Let us now introduce the Hamiltonian density

$$\mathfrak{H}=\sum_{i=1}^{n}\pi_i\,\psi_{it}-\mathfrak{L} \tag{2.34}$$

and the total Hamiltonian:

$$H=\int \mathfrak{H}\,dV. \tag{2.35}$$

We shall also suppose that the condition of the Jacobian of the system (2.32) being different from zero holds, i. e.

$$\left|\frac{\partial \pi_i}{\partial \psi_{jt}}\right|=\left|\frac{\partial^2 \mathfrak{L}}{\partial \psi_{it}\,\partial \psi_{jt}}\right| \neq 0 \tag{2.36}$$

so that the time derivatives ψ_{it} can be determined and substituted by π_i. Thus $\mathfrak{H}$ becomes of the form:

$$\mathfrak{H}=\mathfrak{H}\,(\psi_i, \psi_{ij}, \pi_i, x_j, t) \tag{2.37}$$

and H becomes a functional:

$$H=H\,[\psi_i, \pi_i, t]. \tag{2.38}$$

It should be noted here, too, that the arguments of the functionals in the field theory coincide with the arguments of the corresponding functions in the mechanics of the particle system.

The transformed element of action (2.29), in view of (2.32) and (2.34), can now be written in the form:

$$\Phi = \int \left(\sum_{i=1}^{n} \pi_i \, d\psi_i + \sum_{i=0}^{n} 0, \, d\pi_i - \mathfrak{H} \, dt \right) dV. \tag{2.39}$$

In accordance with (2.38), the role of φ_i is here played by ψ_i, π_i and t. With (2.35), the last equality becomes:

$$\Phi = \int \sum_{i=1}^{n} \pi_i \, d\psi_i \, dV - H \, dt \tag{2.40}$$

since dt is independent of the coordinates.

Functional Pfaff equations (2.12) corresponding to ψ_i are now:

$$d\pi_i = \frac{\delta \Phi}{\delta \psi_i} \qquad (i = 1, 2, \ldots, n) \tag{2.41}$$

and, since, in view of (2.40)

$$\frac{\delta \Phi}{\delta \psi_i} = \frac{\delta}{\delta \psi_i} (-H \, dt) = -\frac{\delta H}{\delta \psi_i} \, dt$$

we obtain

$$d\pi_i = -\frac{\delta H}{\delta \psi_i} \, dt$$

and finally:

$$\frac{d\pi_i}{dt} = -\frac{\delta H}{\delta \psi_i} \qquad (i = 1, 2, \ldots, n). \tag{2.42}$$

Similarly, the functional Pfaff equations corresponding to the variables π_i are:

$$d0 = \frac{\delta \Phi}{\delta \pi_i} \tag{2.43}$$

and, since according to (1.14) we have:

$$\frac{\delta}{\delta \pi_i} \int \sum_{l=1}^{n} \pi_l \, d\psi_l \, dV = \frac{\partial}{\partial \pi_i} \left(\sum_{l=1}^{n} \pi_l \, d\psi_l \right) = \sum_{l=1}^{n} \frac{\partial \pi_l}{\partial \pi_i} d\psi_l = d\psi_i$$

it follows:

$$\frac{\delta \Phi}{\delta \pi_i} = d\psi_i - \frac{\delta H}{\delta \pi_i} \, dt,$$

and the considered equations become:

$$0 = d\psi_i - \frac{\delta H}{\delta \pi_i} \, dt$$

or, finally,

$$\frac{d\psi_i}{dt} = \frac{\delta H}{\delta \pi_i} \qquad (i = 1, 2, \ldots, n). \tag{2.44}$$

The obtained equations (2.42) and (2.44) are *Hamilton equations,* and they determine the state of the field. The functional Pfaff equation for t yields an identity only.

Poisson Brackets. — Let us consider any functional in the form of a volume integral, depending on the arguments of the Hamiltonian, i. e.:

$$F=\int \mathfrak{F}\, dV=F[\psi_i, \pi_i, t]. \tag{2.45}$$

Its differential with respect to time as a parameter, in view of (1.25) is:

$$dF=\int\left(\sum_{i=1}^{n} \frac{\delta F}{\delta \psi_i} d\psi_i+\sum_{i=1}^{n} \frac{\delta F}{\delta \pi_i} d\pi_i+\frac{\delta F}{\delta t} dt\right) dV \tag{2.46}$$

Furthermore, according to (1.14) we have:

$$\int \frac{\delta F}{\delta t} dV=\int \frac{\partial \mathfrak{F}}{\partial t} dV=\frac{\partial}{\partial t} \int \mathfrak{F}\, dV$$

i. e.

$$\int \frac{\delta F}{\delta t} dV=\frac{\partial F}{\partial t} \tag{2.47}$$

so that it follows:

$$\frac{dF}{dt}=\int \sum_{i=1}^{n}\left(\frac{\delta F}{\delta \psi_i} \frac{d\psi_i}{dt}+\frac{\delta F}{\delta \pi_i} \frac{d\pi_i}{dt}\right) dV+\frac{\partial F}{\partial t}. \tag{2.48}$$

In view of the Hamilton equations (2.42) and (2.44) we can further write:

$$\frac{dF}{dt}=\int \sum_{i=1}^{n}\left(\frac{\delta F}{\delta \psi_i} \frac{\delta H}{\delta \pi_i}-\frac{\delta F}{\delta \pi_i} \frac{\delta H}{\delta \psi_i}\right) dV+\frac{\partial F}{\partial t},$$

or, if the symbol:

$$[F, G]=\int \sum_{i=1}^{n}\left(\frac{\delta F}{\delta \psi_i} \frac{\delta G}{\delta \pi_i}-\frac{\delta F}{\delta \pi_i} \frac{\delta G}{\delta \psi_i}\right) dV \tag{2.49}$$

is introduced:

$$\frac{dF}{dt}=[F, G]+\frac{\partial F}{\partial t}. \tag{2.50}$$

The expression (2.49) presents the *Poisson brackets* of the two functionals F and G, and (2.50), written by means of it, expresses the total time dependance of any functional of the type (2.45).

Consider now the volume integrals of the quantities ψ_i and π_i, and denote them by q_i and p_i respectively, i. e.:

$$q_i=\int \psi_i\, dV; \qquad p_i=\int \pi_i\, dV \qquad (i=1, 2, \ldots, n). \tag{2.51}$$

They are functionals of the form (2.45). Applying (2.50) to their total time derivatives, and bearing in mind that they are not explicitly dependent on time, we easily obtain:

$$\frac{dq_i}{dt}=[q_i,\ H],\qquad (i=1,\ 2,\ldots,\ n) \tag{2.52}$$

$$\frac{dp_i}{dt}=[p_i,\ H],\qquad (i=1,\ 2,\ldots,\ n). \tag{2.53}$$

These relations represent the Hamilton equations in the Poisson brackets form.

Generalized Pfaff—Bilimović principle. — Having thus established that the differential equations of a field can be obtained as functional Pfaff equations for the transformed element of action as the functional Pfaffian, we are in position now to formulate the following general principle of theoretical physics:

Physical phenomena, describeable by a Lagrangian, evovle according to functional Pfaff equations, if the element of action transformed to the canonical form is taken as the functional Pfaffian.

We shall call this general principle of theoretical physics the *generalized Pfaff—Bilimović principle;* quite similarly to the Hamilton principle, it unites various branches of theoretical physics into a unique entity.

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