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ON A MERCERIAN THEOREM AND ITS APPLICATION TO THE EQUICONVERGENCE OF CESÀRO AND RIESZ TRANSFORMS

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§ **1. Introduction.** Let Σu_k be a series of complex terms, with partial sum

$$s_n = \sum_0^n u_k .$$

Let us consider the *Cesàro*, continuous and discontinuous *Riesz* transforms of order $r>o$, i. e.,

$$C_n^r\{s_n\} = \frac{\sum_0^n \binom{n-k+r}{n-k} u_k}{\binom{n+r}{n}} = \frac{S_n^r}{\binom{n+r}{n}},$$

$$R_x^r\{s_n\} = x^{-r} \sum_{k \leqslant x} (x-k)^r u_k ,$$

and

$$R_{[x]}^r\{s_n\} .$$

We say that two of these transforms are equiconvergent (see *R. G. Cooke* [1, p. 97]) for a sequence $\{s_n\}$, whenever

$$\lim_{n=\infty} \Big\{ C_n^r\{s_n\} - R_n^r\{s_n\} \Big\} = 0, \tag{1}$$

or

$$\lim_{x=\infty} \Big\{ C_{[x]}^r\{s_n\} - R_x^r\{s_n\} \Big\} = 0, \tag{2}$$

or

$$\lim_{x=\infty} \Big\{ R_x^r\{s_n\} - R_{[x]}^r\{s_n\} \Big\} = 0. \tag{3}$$

The equiconvergence of *Cesàro* and continuous *Riesz* mean was already discussed by *Riesz* (see *E. W. Hobson* [2, p. 96]), who proved that $S_n^k = O(n^r)$, where k is the integer satisfying $r-1 \leqslant k < r$, is a sufficient condition for (2).

Later, *R. F. Cooke* ([1, p. 108] and [3]) showed that $s_n = O(1)$ is a sufficient condition that (1) be satisfied for every $r>o$, and that (see [1, p. 112] and [4])

$$s_n = o(n^r), \qquad 0<r<1,$$
$$s_n = o(n), \qquad 1 \leqslant r,$$

is the „best" sufficient condition to (3).

Finally, *R. P. Agnew* [5] has shown that a sufficient condition that (1) and (2) be satisfied for every $r > 0$, is given by

$$\lim_{n=\infty} u_n = 0$$

or by

$$\sum_0^n k u_k = O(n), \quad n \to \infty .$$

The purpose of this paper is to chow that the necessary and sufficient condition for the equiconvergence of *Cesàro*, discontinuous and continuous *Riesz* transforms, is that the sequence $\{u_n\}$ be summable *Cesàro* of order r to zero i. e.,

$$\lim_{n=\infty} C_n^r \{u_n\} = 0; \tag{4}$$

in other terms,

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) .$$

It is obvious that this result contains all the above results. We shall give here only the proof of $(1) \Leftrightarrow (4)$ (theorem 2). This proof is based on a *Mercerian* theorem (theorem 1). The proof of $(3) \Leftrightarrow (4)$ is analoque and is based on a *Mercerian* theorem[1] of *R. Rado* [6]. This of $(2) \Leftrightarrow (4)$ is obvious, because $(4) \Rightarrow (1)$ and (3) shows that $(4) \Rightarrow (2)$, and from $(2) \Rightarrow (1)$ and $(1) \Rightarrow (4)$ it follows that $(2) \Rightarrow (4)$.

§ 2. A Mercerian theorem. In this paragraph we prove the following theorem.

Theorem 1. Let $\{p_n\}$ be a sequence of complex numbers, $p_0 \neq 0$, and let $\{\overset{*}{p}_n\}$ be defined by

$$p_0 \overset{*}{p}_0 = 1, \quad \sum_{k=0}^n p_{n-k} \overset{*}{p}_k = 0, \quad n = 1, 2, \ldots .$$

Suppose that

$$\sum_0^\infty |p_k| < \infty , \tag{5}$$

$$\sum_0^\infty |\overset{*}{p}_k| < \infty , \tag{6}$$

and that the triangular matrix

$$\{\alpha_{n,k}\}, \quad \alpha_{n,k} = 0, \quad k > n,$$

satisfies the conditions

$$\lim_{n=\infty} \sum_{k=0}^n |\alpha_{n,k+1} - \alpha_{n,k}| = 0, \tag{7}$$

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[1] If $\{\delta_{n,k}\}$ is a permanent triangular matrix, and if

$$\liminf_{n=\infty} |\delta_{n,n}| > \limsup_{n=\infty} \sum_{k=0}^{n-1} |\delta_{n,k}| ,$$

then the transform

$$\delta = \sum_{k=0}^n \delta_{n,k} t_k ,$$

is a *Mercerian* transform.

and

$$(8)\qquad \limsup_{n=\infty} \sum_{k=0}^{n} |\alpha_{n,k}| < \left| \sum_{k=0}^{\infty} p_k \right|.$$

Then, the matrix $\{p_{n-k}+\alpha_{n,k}\}$ defines a *Mercerian* transform; in other words, if we write

$$\sigma_n = \sum_{k=0}^{n} \{p_{n-k}+\alpha_{n,k}\} s_k, \quad n=0, 1, 2, \dots ,$$

we have

$$s_n \to 0 \Leftrightarrow \sigma_n \to 0, \quad n\to\infty.$$

Proof. From (5), (8) and (7) follows that

$$s_n \to 0 \Rightarrow \sigma_n \to 0, \quad n\to\infty$$

because

$$(9)\qquad |\alpha_{n,k}| \leq \sum_{i=k}^{n} |\alpha_{n,i+1}-\alpha_{n,i}| \to 0 \text{ as } n\to\infty, \forall 0\leq k\leq n.$$

To prove the converse, put

$$\gamma_{n,k} = \sum_{j=0}^{n-k} \alpha_{n,k+j} p_j^*,$$

and show at first that (6), (7) and (8) implies

$$(10)\qquad \limsup_{n=\infty} \sum_{k=0}^{n} |\gamma_{n,k}| = \left| \sum_{j=0}^{\infty} p_j^* \right| \limsup_{n=\infty} \sum_{k=0}^{n} |\alpha_{n,k}|.$$

Indeed

$$\sum_{k=0}^{n} |\gamma_{n,k}| = \sum_{k=0}^{n} \left| \alpha_{n,k} \sum_{j=0}^{n} p_j^* + \sum_{j=1}^{n} (\alpha_{n,k+j}-\alpha_{n,k}) p_j^* \right|,$$

hence

$$(11)\qquad \begin{aligned} \sum_{k=0}^{n} |\gamma_{n,k}| - \sum_{k=0}^{n} |\alpha_{n,k}| \left| \sum_{j=0}^{n} p_j^* \right| &\leq \sum_{k=0}^{n} \left| \sum_{j=1}^{n} (\alpha_{n,k+j}-\alpha_{n,k}) p_j^* \right| \\ &\leq \sum_{j=1}^{n} |p_j^*| \sum_{k=0}^{n} |\alpha_{n,k+j}-\alpha_{n,k}|. \end{aligned}$$

By (7) and (9) we see that

$$\sum_{k=0}^{n} |\alpha_{n,k+j}-\alpha_{n,k}| \to 0 \quad \text{as} \quad n\to\infty, \quad \text{for} \quad j=1, 2, \dots ,$$

and as

$$\sum_{k=0}^{n} \alpha_{n,k+j}-\alpha_{n,k}| \leq 2 \sum_{k=0}^{n} |\alpha_{n,k}|,$$

it follows from (8) and (6) that the right side of (11) tends to zero, as $n\to\infty$; hence (10) is true.

Now, by (5) and (6) it follows that the transform defined by

$$(12)\qquad t_n = \sum_{k=0}^{n} p_{n-k} s_k$$

is a *Mercerian* transform. Therefore, it is sufficient to show that the trans form obtained by expressing σ_n as the transform of t_n is *Mercerian.*

By (12)

$$s_n = \sum_{k=0}^{n} p^*_{n-k}\, t_k ,$$

which gives

(13) $$\sigma_n = t_n + \sum_{k=0}^{n} \gamma_{n,k}\, t_k ,$$

where

$$\gamma_{n,k} = \sum_{j=0}^{n-k} \alpha_{n,k+j}\, p^*_j , \quad n = 0, 1, \ldots \text{ and } 0 \leqslant k \leqslant n.$$

From a theorem of *R. Rado* [6][2], it will follow that (13) is a *Mercerian* transform, if we show that

$$\lim_{n=\infty} \gamma_{n,n} = 0 \quad \text{and} \quad \limsup_{n=\infty} \sum_{k=0}^{n} |\gamma_{n,k}| < 1.$$

But from (9) follows that $\lim_{n=\infty} \gamma_{n,n} = 0$, and from (10) and (8),

$$\limsup_{n=\infty} \sum_{k=0}^{n} |\gamma_{n,k}| = \left| \sum_{j=0}^{\infty} p^*_j \right| \limsup_{n=\infty} \sum_{k=0}^{n} |\alpha_{n,k}| < \left| \sum_{j=0}^{\infty} p_j \sum_{j=0}^{\infty} p^*_j \right| = 1,$$

which proves the theorem.

§ 3. The necessary and sufficient condition for the equiconvergence of Cesàro and discontinuous Riesz transforms.

Theorem 2. Let $r > 0$ be a fixed number. Then $C^r_n\{s_n\}$ and $R^r_n\{s_n\}$ are equiconvergent, in the sense of (1), if and only if the sequence $\{u_n\}$ is summable *Cesàro* of order r to zero; in other terms, setting

$$\Delta_r(n) = C^r_n\{s_n\} - R^r_n\{s_n\} ,$$

we have

(14) $$\Delta_r(n) \to 0 \Leftrightarrow C^r_n\{u_n\} \to 0, \quad n \to \infty .$$

For the proof of theorem 2, we use two lemmas, which we shall establish at first.

Lemma 1. For every $r > 0$, the function $\Psi_r(z)$ defined by

(15) $$(1-z)\Psi_r(z) = \Gamma(r+1) - (1-z)^{r+1} \sum_{n=1}^{\infty} n^r z^n ,$$

where we choose the principal branch of $(1-z)^r$, is

(i) regular in the whole plan cut along the real axis from $z = 1$, to $z = +\infty$,

(16) (ii) $\Psi_r(z) = P_k(z) + O(|1-z|^r)$ as $|z-1| \to 0,$

where $P_k(z)$ iz a polynomial of order k, and k is the integer satisfying $r-1 \leqslant k < r$.

Proof. Setting

$$f(t) = (1-z)^{r+1} e^{t \lg z}\, t^r, \quad |z| < 1,$$

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[2] See l. c. 1).

where we choose for $\lg z$ and for $(1-z)^r$ the principal branch, and establishing *Poisson*'s formula (see [7], [8, p. 39—45] and [9, p. 68])

$$\sum_{n=1}^{\infty} f(n) = \sum_{\mu=-\infty}^{+\infty} \int_0^{\infty} f(t)\, e^{2\mu\pi i t}\, dt,$$

we obtain

$$(1-z)^{r+1} \sum_{n=1}^{\infty} n^r z^n = \Gamma(r+1) \sum_{\mu=-\infty}^{+\infty} \left(\frac{z-1}{\lg z + 2\mu\pi i}\right)^{r+1} \tag{17}$$

for $|z|<1$, where we take the principal branch of $\lg z$ and of

$$\left(\frac{z-1}{\lg z + 2\mu\pi i}\right)^{r+1}.$$

From (17), by analytic continuation, it follows the first affirmation of lemma 1.

We see also that if the constants γ_ν are defined by

$$\Gamma(r+1)\left(\frac{z-1}{\lg z}\right)^{r+1} = \sum_{\nu=0}^{\infty} \gamma_\nu (1 \quad z)^\nu, \quad |z-1|<1, \tag{18}$$

then

$$(1-z)^{r+1} \sum_{n=1}^{\infty} n^r z^n = \sum_{\nu=0}^{\infty} \gamma_\nu (1-z)^\nu + \Gamma(r+1) \sum_{\mu=-\infty}^{+\infty}{}' \left(\frac{z-1}{\lg z + 2\mu\pi i}\right)^{r+1},$$

where the dash' indicates that the term $\mu=0$ is omitted from the summation. As the last series in the right side is absolutely and uniformely convergent for every z, $z=|z|e^{i\theta}$, $|\theta|\leqslant\pi$, we have

$$(1-z)^{r+1} \sum_{n=1}^{\infty} n^r z^n = \gamma_0 + \gamma_1(1-z) + \cdots + \gamma_{k+1}(1-z)^{k+1} + O(|1-z|^{r+1}), \tag{19}$$

when $|z-1|\to 0$ and $r-1\leqslant k<r$.

From (18) we see that

$$\gamma_0 = \Gamma(r+1) \quad \text{and} \quad \gamma_1 = -\frac{1}{2}\Gamma(r+2). \tag{20}$$

Now, we see immediately that (16) follows from (15), (19) and (20), and that

$$\Psi_r(z) \to \frac{1}{2}\Gamma(r+2) \text{ as } |z-1|\to\theta. \tag{21}$$

Lemma 2. Setting

$$\Psi_r(z) = \sum_{n=0}^{\infty} p_n z^n \tag{22}$$

and

$$\frac{1}{\Psi_r(z)} = \sum_{n=0}^{\infty} p_n^* z^n,$$

the two series $\sum_{n=0}^{\infty} p_n$ and $\sum_{n=0}^{\infty} p_n^*$ are absolutely convergent.

Proof. By *Cauchy*'s formula, we hawe, for $n \geqslant k+1$,

$$p_n = \frac{1}{2\pi i} \int_C \frac{\Psi_r(\zeta)}{\zeta^{n+1}} d\zeta = \frac{1}{2\pi i} \int_C \frac{\Psi_r(\zeta) - P_k(\zeta)}{\zeta^{n+1}} d\zeta,$$

and choosing for C a circle of radius $R>1$, avoiding the point 1 by a loop, we obtain, taking (16) into consideration, that

$$|p_n| \leqslant \frac{M_1}{R^n} + M_2 \int_1^R \frac{(t-1)^r}{t^{n+1}} dt = O\left(\frac{1}{n^{r+1}}\right), \quad \text{as} \quad n \to \infty. \tag{23}$$

Thus the series $\sum_0^\infty p_n$ is absolutely convergent for every $r>0$, and by (21) and (22) we get

$$\sum_{n=0}^\infty p_n = \frac{1}{2}\Gamma(r+2). \tag{24}$$

To prove the absolute convergence of $\sum_{n=0}^\infty p_n^*$, we show first that for every $r>0$,

$$\Psi_r(z) \neq 0 \quad \text{for} \quad |z| \leqslant 1 \quad \text{and} \quad z \neq 1. \tag{25}$$

Indeed, when $0<r\leqslant 1$, the coefficients

$$c_n = \frac{\Gamma(r+n+1)}{n!} - n^r$$

of the series

$$\frac{\Gamma(r+1)}{(1-z)^{r+1}} - \sum_{n=1}^\infty n^r z^n = \sum_{n=0}^\infty \left\{\frac{\Gamma(r+n+1)}{n!} - n^r\right\} z^n,$$

decrease and tend to zero as $n\to\infty$ and the affirmation follows from *Kakeya*'s theorem [10][3].

From the *Stirling* formula it follows that

$$\frac{\Gamma(r+n+1)}{n!} = n^r + 0(n^{r-1}) \quad \text{as} \quad n \to \infty, \tag{26}$$

which shows that $c_n \to 0$ as $n\to\infty$, for $0<r<1$.

To prove that $c_{n-1} \geqslant c_n$, which is equivalent to

$$n^r - (n-1)^r \geqslant \frac{r\,\Gamma(r+n)}{n!}, \qquad n=1, 2, \ldots,$$

we use the following fact: it $\{a_n\}$ and $\{b_n\}$ are two sequences satisfying

$$\frac{a_{n-1}}{a_n} \geqslant \frac{b_{n-1}}{b_n}, \qquad n=1, 2, \ldots, \tag{27}$$

and

$$\lim_{n=\infty} \frac{a_n}{b_n} = 1,$$

then

$$a_n \geqslant b_n, \qquad n=1, 2, \ldots.$$

[3] If $c_{n-1} \geqslant c_n \geqslant 0$ for $n=1, 2, \ldots$, and if $c_{n-1} > c_n > 0$ holds for at least one integer n, then

$$\left|\sum_{n=0}^\infty c_n z^n\right| > 0, \quad \text{for} \quad |z| \leqslant 1.$$

Setting

$$a_n = (n+1)^r - n^r \quad \text{and} \quad b_n = \frac{r\,\Gamma(r+n+1)}{(n+1)!},$$

we see by (26) that $a_n/b_n \to 1$ as $n \to \infty$.

To prove (27), which is

$$\frac{n^r - (n-1)^r}{(n+1)^r - n^r} \geqslant \frac{n+1}{n+r}, \tag{28}$$

we divide every expresion in the right side of (28) by n, and every expression in the left side by n^r, and setting $x = \frac{1}{n}$, (28) is reduced to

$$(1+rx)\{1-(1-x)^r\} \geqslant (1+x)\{(1+x)^r - 1\},$$

which is verified for $0<r<1$ and $0<x<1$. Hence (27) is proved. When $r=1$, we see by (15) that $\Psi_r(z)=1$. When $r>1$, we have also

$$\Gamma(r+1) > \left| (1-z)^{r+1} \sum_{n=1}^{\infty} n^r z^n \right| \quad \text{for } |z| \leqslant 1,\ z \neq 1.$$

Indeed, by *Cauchy*'s maximum modulus theorem it is enougt to show that this inequality holds for $z = e^{2\theta\pi i}$, $0<\theta<1$, and as by (17) we have

$$\left| \sum_{n=1}^{\infty} n^r z^r \right| = \frac{\Gamma(r+1)}{(2\pi)^{r+1}} \left| \sum_{\mu=-\infty}^{+\infty} \frac{1}{(\mu+\theta)^{r+1}} \right|, \quad z = e^{2\theta\pi i},\ 0<\theta<1,$$

the inequality is reduced to

$$\left(\frac{\pi}{\sin \theta\pi}\right)^{r+1} > \sum_{\mu=-\infty}^{+\infty} \frac{1}{|\mu+\theta|^{r+1}}, \quad r>1,\ 0<\theta<1.$$

As

$$\frac{\pi^2}{\sin^2 \theta\pi} = \sum_{\mu=-\infty}^{+\infty} \frac{1}{(\mu+\theta)^2},$$

the last inequality is reduced to

$$\left(\sum_{\mu=-\infty}^{+\infty} \frac{1}{(\mu+\theta)^2} \right)^{1/2} > \left(\sum_{\mu=-\infty}^{+\infty} \frac{1}{|\mu+\theta|^{r+1}} \right)^{\frac{1}{r+1}},$$

which is verified for $r>1$, since for $a_\mu > 0$, $(\sum a_\mu{}^t)^{1/t}$ decreases when t increases (see [11, p. 28]).

Hence (25) is verified, which proves that $1/\Psi_r(z)$ is regular for $|z| \leqslant 1$, $z \neq 1$, and according to (16),

$$\frac{1}{\Psi_r(z)} = \frac{1}{P_k(z) + 0(|1-z|^r)} = Q_k(z) + O(|1-z|^r), \quad \text{as } |z-1| \to 0,$$

where Q is a polynomial of order k; it follows for the same reasons as before, that

$$p_n^* = O\left(\frac{1}{n^{r+1}}\right), \tag{29}$$

which proves the assertion.

Proof of theorem 2. As

$$C_n^r\{u_n\}=\frac{r}{n+r}C_n^{r-1}\{s_n\}$$

and as

$$\frac{r}{n+r}C_n^{r-1}\{s_n\}\to 0 \Leftrightarrow n^{-r}S_n^{r-1}\to 0, \quad \forall r>0,$$

(14) is reduced to

$$\Delta_r(n)\to 0 \Leftrightarrow n^{-r}S_n^{r-1}\to 0, \quad n\to\infty, \quad \forall r>0.$$

Multiplying (15) by $(1-z)^{-r-1}\sum_{n=0}^{\infty}u_n z^n$, we get by (22)

$$\Gamma(r+1)S_n^r-n^r R_n^r\{s_n\}=\sum_{k=0}^{n}p_{n-k}S_k^{r-1},$$

and so

$$\Delta_r(n)=n^{r-2}\sum_0^n p_{n-k}S_k^{r-1}+\left(\frac{1}{\binom{n+r}{n}}-\frac{\Gamma(r+1)}{n^r}\right)S_n^r.$$

Since

$$S_n^r=\sum_{k=0}^{n}S_k^{r-1},$$

by setting

$$\rho_0=S_0^{r-1} \quad \text{and} \quad \rho_n=n^{-r}S_n^{r-1} \quad \text{for} \quad n\geqslant 1,$$

we get

(30)
$$\Delta_r(n)=\sum_{k=0}^{n}(p_{n-k}+\alpha_{n,k})\rho_k,$$

where

$$\alpha_{n,0}=\left(\frac{1}{\binom{n+r}{n}}-\frac{\Gamma(r+1)}{n^r}\right)-(1-n^{-r})p_n,$$

and

$$\alpha_{n,k}=\left(\frac{1}{\binom{n+r}{n}}-n^{-r}\Gamma(r+1)\right)k^r-(1-n^{-r}k^r)p_{n-k}, \quad 1\leqslant k\leqslant n$$

and it is sufficient to show that the transform defined by (30) satisfies the conditions of theorem 1.

But, from (23) and (29) it follows that conditions (5) and (6) are satisfied. Besides,

$$\sum_{k=0}^{n}|\alpha_{n,k}|\leqslant\Gamma(r+1)\left(n^{-r}-\frac{n!}{\Gamma(r+n+1)}\right)\left(1+\sum_1^n k^r\right)+(1-n^{-r})|p_n|$$
$$+\sum_{k=1}^{n}(1-n^{-k}k^r)|p_{n-k}|=$$
$$\leqslant\Gamma(r+1)\left(n^{-r}-\frac{n!}{\Gamma(n+r+1)}\right)\sum_1^n k^r+o(1)=\frac{r}{2}\Gamma(r+1)+o(1)$$

and so by (24),

$$\limsup_{n=\infty}\sum_{k=0}^{n}|\alpha_{n,k}|<\left|\sum_0^{\infty}p_k\right|,$$

which shows that the condition (8) is satisfied.

Finally,

$$\sum_{k=0}^{n} |\alpha_{n,\,k+1}-\alpha_{n,\,k}| = (1-n^{-r})\,|p_n - p_{n-1}| + \left(\Gamma(r+1) - n^r \Big/ \binom{n+r}{n}\right)$$

$$+\sum_{k=1}^{n-1}\left|\left(1 \Big/ \binom{n+r}{n} - n^{-r}\,\Gamma(r+1)\right)((k+1)^r - k^r) - (1-n^{-r}(k+1)^r)\,p_{n-k-1}\right.$$

$$\left.+(1-n^{-r}k^r)\,p_{n-k}\right| \leqslant 2\left(\Gamma(r+1) - n^r \Big/ \binom{n+r}{n}\right) + 2\sum_{k=0}^{n}(1-n^{-r}k^r)\,|p_{n-k}| = o(1),$$

which shows that the condition (7) is satisfied.. Thus, theorem 2 is a special case of theorem 1.

REFERENCES

[1] R. G. Cooke: *Infinite matrices and sequences space.* Macmillan, 1950.

[2] E. W. Hobson: *The theory of functions of a real variable.* Cambridge, second edition, 1926, vol. II.

[3] R. G. Cooke: *On mutal and regular T-limits.* Proc. Lond. Math. Soc. (2) **41,** 1936, p. 114 and pp. 117—119.

[4] R. G. Cooke: *An extension of some recent results on mutual consistency of regular T-limits.* Jour. Lond. Math. Soc. **12,** 1937, pp. 98—105.

[5] R. P. Agnew: *Equiconvergence of Cesàro and Riesz transforms of series.* Duke Math. Jour. **22,** № 3, 1955, pp. 451—460.

[6] R. Rado: *Some elementary Tauberian theorems.* Quart. Jour. of Math. **9,** 1938, pp. 274—282.

[7] J. R. Wilton: *A proof of Poisson's summation formula.* Jour. Lond. Math. Soc. **5,** 1930, pp. 276—279.

[8] S. Bochner: *Lectures on Fourier integrals.* Princeton. 1959. pp. 39—45.

[9] A. Zygmund: *Trigonometric series.* Cambridge. 1959, Vol. I, p. 68.

[10] S. Kakeya: *On the limits of the roots of ar algebraic equation with positive coefficients.* Tôhoku Math. Jour. 2, 1912, pp. 140—142.

[11] G. H. Hardy, J. E. Littlewood and G. Polya: *Inequalities.* Cambridge 1952.