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ON A CONNEXION BETWEEN LEGENDRE'S FUNCTIONS

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It is known that *Legendre*'s complete normal elliptic integrals of the first and second kind can be expressed also in terms of *Gauss*'s hypergeometric function as follows

$$(1)\qquad F(\alpha, \beta, \gamma; x) = 1 + \sum_{\nu=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+\nu-1)\beta(\beta+1)\cdots(\beta+\nu-1)}{\nu!\,\gamma(\gamma+1)\cdots(\gamma+\nu-1)} x^{\nu},$$

namely

$$(2)\qquad K=\frac{\pi}{2}F\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right), \qquad E=\frac{\pi}{2}F\left(-\frac{1}{2}, \frac{1}{2}, 1; k^2\right),$$

where K and E are the above mentioned integrals:

$$(3)\qquad K=\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \qquad E=\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\varphi}\, d\varphi.$$

On the other hand, *Legendre*'s functions of the first kind of index n can be expressed in the form

$$(4)\qquad P_n(\lambda) = F\left(-n, n+1, 1; \frac{1-\lambda}{2}\right)$$

(n is not the negative whole number).

Gonzáles has shown in [1] that

$$(5)\qquad K=\frac{\pi}{2}P_{-\frac{1}{2}}(\lambda), \qquad E=\frac{\pi}{4}\left[P_{-\frac{1}{2}}(\lambda)+P_{\frac{1}{2}}(\lambda)\right]$$

where $(1-\lambda)/2=k^2$, and derived also formulae for complementary integrals K' and E'

$$(6)\qquad K'=\frac{\pi}{2}P_{-\frac{1}{2}}(-\lambda), \qquad E'=\frac{\pi}{4}\left[P_{-\frac{1}{2}}(-\lambda)+P_{\frac{1}{2}}(-\lambda)\right],$$

where K' and E' have the same form as in (1), only instead of the module $k \in [0, 1]$ there is the complementary module k' connected with the module k of the equation $k^2+k'^2=1$.

In this note, I profit by *Gonzáles* results and obtain an interesting connexion between *Legendres's* functions of the index 1/2 and —1/2, namely

$$(I) \qquad P_{\frac{1}{2}}(\lambda)P_{-\frac{1}{2}}(-\lambda)+P_{-\frac{1}{2}}(\lambda)P_{\frac{1}{2}}(-\lambda)=4/\pi.$$

Proof. The lefthand side of the equation (I) can be expressed in the form

$$P_{-\frac{1}{2}}(-\lambda)P_{-\frac{1}{2}}(\lambda)+P_{-\frac{1}{2}}(-\lambda)P_{\frac{1}{2}}(\lambda)+P_{-\frac{1}{2}}(\lambda)P_{-\frac{1}{2}}(-\lambda)+P_{-\frac{1}{2}}(\lambda)P_{\frac{1}{2}}(-\lambda)$$
$$-2P_{-\frac{1}{2}}(\lambda)P_{-\frac{1}{2}}(-\lambda)$$

or

$$P_{-\frac{1}{2}}(-\lambda)\left[P_{-\frac{1}{2}}(\lambda)+P_{\frac{1}{2}}(\lambda)\right]+P_{-\frac{1}{2}}(\lambda)\left[P_{-\frac{1}{2}}(-\lambda)+P_{\frac{1}{2}}(-\lambda)\right]$$
$$-2P_{-\frac{1}{2}}(\lambda)P_{-\frac{1}{2}}(-\lambda).$$

On the basic of *Gonzáles* formulae (2) and (3), the first factor of the first term has the value $2K'/\pi$, the second factor the value $4E/\pi$; the first factor of the second term has the value $2K/\pi$, the second factor the value $4E'/\pi$. On the basic of these formulae, the last expression can be expressed in the form

$$\frac{8}{\pi^2}(K'E+KE'-KK').$$

But, on the basic of the clasical *Legendre's* relation for elliptic integrals of the first and second kind

$$K'E+KE'-KK'=\pi/_2,$$

at once follows the righthand side of the equation (I) which was to be shown.

REFERENCE

[1] M. O. Gonzáles: *Elliptic Integrals in Terms of Legendre Polynomials,* Proc. Glasgov Math. Ass., Vol. II (1954), pp. 97—99.