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AN ANALOGY BETWEEN THE CLASSICAL AND THE BORN RELATIVISTIC RIGID BODY

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Let us consider the system of particles C_{ξ^i} $(i = 1, 2, 3)$, where the ξ^i are *Lagrange*'s parameters characterizing individual particles of the system.

The classical definition of the rigid body: „The system of particles is said to be a rigid body if the distance between simultaneous positions of any two particles of the system is constant in the time and depends only on the choice of the two particles", is inconsistent with the theory of relativity since it is based on the notion of simultaneity which in this theory has not an absolute meaning. Consequently, adopting the definition quoted above, in the theory of relativity it might be only said that the system C_{ξ^i} is rigid with respect to a presigned observer, and the „rigidity" would not be a *natural* property of the system itself, from the point of view of the theory of relativity.

Looking for a property of the system of particles C_{ξ^i} which would be both *covariant* with respect to the relativistic transformations (and, consequently, independent from the observer), and the classical rigid body under the classical approximation of the relativistic results, *Max Born*[1] gave the following definition of relativistic rigid body: „The system of particles C_{ξ^i} is said to be the relativistic rigid body if for any two adjacent particles of the system the interval between the corresponding world lines, orthogonal to those lines, is constant during the motion". The terms „interval" and „orthogonal" used in this definition are to be understood in the sense of the space-time metrics.

Although *Herglotz*[2] and *Noether*[3] demonstrated that the body defined in this way has in the special theory of relativity only three degrees of freedom, all papers on the relativistic rigid body are based on *Born*'s definition, for this is, at present, the only formulated definition.

Studying the motion of *Born*'s rigid body we observed two characteristic properties of that motion. Explaining them we shall follow the notation *Salzman* and *Taub* used in their paper „*Born*-type rigid motion in the relativity"[4]. *Born*'s definition given above is due to the same authors. The Latin subscripts take the values 1, 2, 3, and the Greek ones 1, 2, 3, 4.

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[1] Max Born, *Annalen der Physik*, **30,** 1 (1909).

[2] G. Herglotz, *Annalen der Physik*, **31,** 393 (1909—1910).

[3] F. Noether, *Annalen der Physik*, **31,** 919 (1909—1910).

[4] G. Salzman and A. H. Taub, *Physical Review*, **95,** 1659 (1954).

In a given system of coordinates x^α of the space-time, where $x^4 = ct$, c being the velocity of light and t the time in this system of coordinates, the motion of the system of particles is determined by

$$x^\alpha = x^\alpha(\xi^i, \theta), \tag{1}$$

where θ is any timelike parameter. It is assumed that the equations (1) represent a nonsingular transformation between the coordinates x^α and the coordinates ξ^i, θ. Further, it is supposed that (1) represents, for fixed values of ξ^i, the parametric equations of a timelike line, since by the assumption none of the particles C_{ξ^i} of any system has at any instant the velocity which is equal or greater than that of light. Let $g_{\alpha\beta}$ be the metric tensor of the space-time in the x^α system of coordinates, such that the signature of the form

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

is 2, and that the spacelike interval defined by the form is positive.

Let

$$U^\alpha = \frac{\partial x^\alpha}{\partial \theta}. \tag{2}$$

U^α is a timelike fourvector, its modulus is

$$(-g_{\alpha\beta}\, U^\alpha\, U^\beta)^{\frac{1}{2}}. \tag{3}$$

The fourvector

$$u^\alpha = (-g_{\mu\nu}\, U^\mu\, U^\nu)^{-\frac{1}{2}}\, U^\alpha \tag{4}$$

is the fourvector for which

$$g_{\alpha\beta}\, u^\alpha\, u^\beta = -1, \tag{5}$$

i.e., u^α is the unit four-velocity vector. From (5) follows

$$u_{\alpha;\beta}\, u^\alpha = 0, \tag{6}$$

where

$$u_\alpha = g_{\alpha\beta}\, u^\beta, \tag{7}$$

and $u_{\alpha;\beta}$ is the covariant derivative of the vector u_α with respect to x^β.

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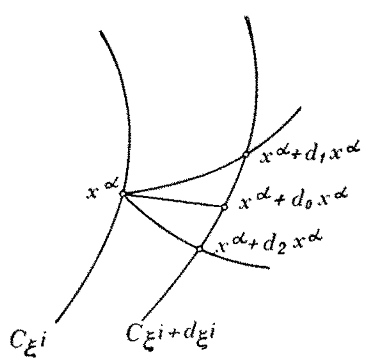

Let us consider now the world lines of two adjacent particles C_{ξ^i} and $C_{\xi^i + d\xi^i}$ (see the figure), and on the world line C_{ξ^i} an event x^α. The set of world lines of the light rays through x^α forms the null hypersurface of the space-time (in the special theory of relativity the null hypercone) through this event. We shall call this hypersurface the null hypersurface of the event x^α. This hypersurface intersects the world line $C_{\xi^i + d\xi^i}$ in two events $x^\alpha + d_1 x^\alpha$ and $x^\alpha + d_2 x^\alpha$. In the middle of the two events is the event $x^\alpha + d_0 x^\alpha$ where

$$d_0 x^\alpha = \frac{d_1 x^\alpha + d_2 x^\alpha}{2}, \tag{8}$$

which is also, in the first approximation, on the world line $C_{\xi^i + d\xi^i}$.

We shall now prove that *the vector d_0x^α is orthogonal to the vector u^α, i. e., to the world line C_{ξ^i}, and that, consequently, it represents an orthogonal interval mentioned in Born's definition.*

From (1), according to (2),

$$dx^\alpha = \frac{\partial x^\alpha}{\partial \xi^i} d\xi^i + \frac{\partial x^\alpha}{\partial \theta} d\theta,$$

i. e.,

(9) $$dx^\alpha = x^\alpha{}_{,i}\, d\xi^i + U^\alpha d\theta.$$

Varying only $d\theta$ (i. e., taking ξ^i, θ, and $d\xi^i$ as fixed) we obtain various events on the world line $C_{\xi^i+d\xi^i}$. Because of

(10) $$d_1x^\alpha = x^\alpha{}_{,i}\, d\xi^i + U^\alpha d_1\theta,$$

and

(11) $$d_2x^\alpha = x^\alpha{}_{,i}\, d\xi^i + U^\alpha\, d_2\theta,$$

and according to (8) and (9),

(12) $$d_0x^\alpha = x^\alpha{}_{,i}\, d\xi^i + U^\alpha \frac{d_1\theta + d_2\theta}{2},$$

where $d_1\theta$ and $d_2\theta$ are solutions of the equation

(13) $$g_{\alpha\beta}\,(x^\alpha{}_{,i}\, d\xi^i + U^\alpha\, d\theta)\,(x^\beta{}_{,j} d\xi^j + U^\beta\, d\theta) = 0.$$

This equation may be written as

(14) $$g_{\alpha\beta}\, U^\alpha\, U^\beta\, (d\theta)^2 + 2g_{\alpha\beta}\, U^\alpha\, x^\beta{}_{,i}\, d\xi^i \cdot d\theta + g_{\alpha\beta}\, x^\alpha{}_{,i}\, x^\beta{}_{,j}\, d\xi^i\, d\xi^j = 0,$$

wherefrom

(15) $$\frac{d_1\theta + d_2\theta}{2} = \frac{g_{\lambda\mu}\, U^\lambda\, x^\mu{}_{,i}\, d\xi^i}{-g_{\sigma\tau}\, U^\sigma\, U^\tau},$$

so that (12) becomes

$$d_0\, x^\alpha = \left(x^\alpha{}_{,i} + U^\alpha \frac{g_{\lambda\mu}\, U^\lambda\, x^\mu{}_{,i}}{-g_{\sigma\tau}\, U^\sigma\, U^\tau} \right) d\xi^i,$$

or, referring to (4),

(16) $$d_0\, x^\alpha = (x^\alpha{}_{,i} + u^\alpha\, u_\lambda\, x^\lambda{}_{,i})\, d\xi^i.$$

The scalar product of the vectors u^α and $d_0\, x^\alpha$ is

$$g_{\alpha\beta}\, u^\alpha\, d_0\, x^\beta = (g_{\alpha\beta}\, u^\alpha\, x^\beta{}_{,i} + g_{\alpha\beta}\, u^\alpha\, u^\beta\, u_\lambda\, x^\lambda{}_{,i})\, d\xi^i,$$

so that, because of (5),

(17) $$g_{\alpha\beta}\, u^\alpha\, d_0\, x^\beta = (u_\beta\, x^\beta{}_{,i} - u^\lambda x^\lambda{}_{,i})\, d\xi^i = 0,$$

which proves the statement.

It should be noted that it is always possible to find a system of coordinates with respect to which the events x^α and any event on the world line $C_{\xi^i+d\xi^i}$ between the events $x^\alpha + d_1x^\alpha$ and $x^\alpha + d_2x^\alpha$ are simultaneous, so that the event x^α and each of the mentioned events on the world line $C_{\xi^i+d\xi^i}$

are, after *V. A. Fok*[5], quasisimultaneous. Therefore we shall call $x^\alpha + d_0 x^\alpha$ the mean quasisimultaneous event of the world line $C_{\xi^i + d\xi^i}$ corresponding to the event x^α.

Born's rigid body can be defined now as follows: „The system of particles C_{ξ^i} is said to be a relativistic rigid body, if during the motion the interval between the event on the world line of any particle of the system and the mean quasisimultaneous event corresponding to the former, on the world line of the adjacent particle, remains constant".

We think that this definition of *Born*'s rigid body, based on its just proved property, is more suitable than *Born*'s original definition, because it is closer to its classical analogon.

To explain this, we shall first define some notions of the classical kinematics.

An event in classical kinematics is the quantity determined with four numbers: the three of them x^i determine the position of the point in the space and the fourth, x^4, determines the instant in which the point is considered. Corresponding to each such event there is a point (event) in the four-dimensional space, which we call the classical space-time.

In the classical space-time the admissible transformations

(18) $$\bar{x}^\alpha = \bar{x}^\alpha (x^1, x^2, x^3, x^4),$$

are subject, except to the requirement of nonsingularity, to the condition

(19) $$\frac{\partial \bar{x}^4}{\partial x^i} = 0,$$

which is the consequence of the absolutness of the time in the classical kinematics. This means that the events simultaneous in one system of coordinates x^α are simultaneous in every other admissible system of coordinates $\bar{x}^\alpha$.

Speaking about the metrics of the classical space-time we shall restrict ourselves only to the intervals between events which lie in the same, whatever, hyperplane $x^4 = constant$ and such an interval we shall call, as usually, the distance.

A continuous set of events corresponds to the motion of a particle, and such a motion is thus represented in the classical space-time by a line, which we shall call the classical world line.

The classical kinematics may be regarded as an approximation to the relativistic one, if the velocities of the particles considered are small in comparison with the velocity of light, or, with the same approximation, regarding the velocity of light as infinite in comparison with the velocities considered, i. e. that the head of the light wave is *simultaneously present* in all points of its path.

Regarding the classical kinematics as such an approximation of the relativistic one, the classical analogon to the relativistic null hypersurface of the event x^α is a hyperplane through x^α — the geometric locus of simultaneous events. We shall call it the hyperplane of the event x^α .

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[5] В. А. Фок, *Теория пространства, времени и тяготения*, Гостехиздат, Москва, 1955, стр. 50.

It is obvious now that the event x^α of the classical world line of a particle C_{ξ^i} and the event on the classical world line of the particle $C_{\xi^i + d\xi^i}$, in which the latter intersects the hyperplane of the event x^α, are simultaneous. Hence, the analogy to the relativistic notion of simultaneous event is established.

Now, the classical definition of the rigid body may be formulated as follows: ,,The system of particles C_{ξ^i} is said to be the rigid body, if the distance between the event on the classical world line of any particle of the system and the event on the classical world line of another particle simultaneous to the former remains constant during the motion.

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Born's definition requires that the fundamental form

$$dl^2 = g_{\alpha\beta}\, d_0 x^\alpha\, d_0 x^\beta\,, \tag{20}$$

for fixed values of ξ^i and $d\xi^i$, is independent of the motion, i. e.

$$(dl^2)_{,\theta} = 0. \tag{21}$$

Substituting (16) into (20), and because of (5) the equation (21) becomes

$$[(g_{\alpha\beta} + u_\alpha u_\beta)\, x^\alpha{}_{,i}\, x^\beta{}_{,j}\, d\xi^i\, d\xi^j]_{,\theta} = 0, \tag{22}$$

wherefrom *Salzman* and *Taub* concluded correctly that it must be

$$[(g_{\alpha\beta} + u_\alpha u_\beta)\, x^\alpha{}_{,i}\, x^\beta{}_{,j}]_{,\theta} = 0. \tag{23}$$

From (23) *Salzman* and *Taub*, by identical transformations, obtained the equations

$$D_{\alpha\beta}\, x^\alpha{}_{,i}\, x^\beta{}_{,j} = 0, \tag{24}$$

where

$$D_{\alpha\beta} = u_{\alpha;\beta} + u_{\beta;\alpha} + u_{\alpha;\lambda}\, u^\lambda\, u_\beta + u_{\beta;\lambda}\, u^\lambda\, u_\alpha\,. \tag{25}$$

Changing the dummies the equations (24) can be written in the form

$$u_{\alpha;\beta}\,[x^\alpha{}_{,i}\,(x^\beta{}_{,j} + u^\beta u_\lambda x^\lambda{}_{,j}) + x^\alpha{}_{,j}\,(x^\beta{}_{,i} + u^\beta u_\lambda x^\lambda{}_{,i})] = 0. \tag{26}$$

Multiplying these equations by $d\xi^i\, d\xi^j$ and summing with respect to i and j, and exchanging then the dummies i and j in the second term, we obtain

$$u_{\alpha;\beta}\, x^\alpha{}_{,i}\,(x^\beta{}_{,j} + u^\beta u_\lambda x^\lambda{}_{,j})\, d\xi^i\, d\xi^j = 0,$$

or, because of (16),

$$u_{\alpha;\beta}\, x^\alpha{}_{,i}\, d_0 x^\beta\, d\xi^i = 0. \tag{27}$$

Multiplying (6) with

$$u_\mu\, x^\mu{}_{,i}\, d_0\, x^\beta\, d\xi^i$$

and summing with respect to β we obtain

$$u_{\alpha;\beta}\, u^\alpha\, u_\mu\, x^\mu{}_{,i}\, d_0\, x^\beta\, d\xi^i = 0. \tag{28}$$

Adding (27) to (28) we obtain

$$u_{\alpha;\beta}\,(x^\alpha{}_{,i} + u^\alpha u_\mu x^\mu{}_{,i})\, d\xi^i\, d_0 x^\beta = 0,$$

or, because of (16),

$$u_{\alpha;\beta}\, d_0 x^\alpha\, d_0 x^\beta = 0. \tag{29}$$

The equation (29), written in the form

$$g_{\alpha\beta} d_0\, x^\alpha\,(u^\beta{}_{;\gamma}\, d_0\, x^\gamma) = 0 \tag{30}$$

express the result which may be formulated geometrically as follows: „During the motion of *Born*'s rigid body, the change of the vector u^α by a displacement from the event x^α on the world line C_{ξ^i} to the event $x^\alpha + d_0 x^\alpha$ on the world line $C_{\xi^i + d\xi^i}$, which is in the middle between the events on this world line in which the latter is intersected by the null hypersurface of the event x^α, is orthogonal to the displacement".

This property is equivalent to the definition of *Born*'s rigid body, what easily may be seen deducing (24) from (30) by a reverse process, and consequently, it may be taken as the definition of *Born*'s rigid body.

If we remember the analogy between the displacement, mentioned above, and the line-segment connecting the simultaneous events of two fixed particles of the classical rigid body, an analogous property corresponds to the property of *Born*'s rigid body, given above, of the classical rigid body, which, expressed in the vector form, is

$$(\vec{r}_{C_{\xi^i + d\xi^i}} - \vec{r}_{C_{\xi^i}}) \cdot (\vec{v}_{C_{\xi^i + d\xi^i}} - \vec{v}_{C_{\xi^i}}) = 0,$$

where $\vec{r}$ is the radius vector of the particle, $\vec{v}$ its velocity, and "." the scalar product.