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ON A CONVERGENCE THEOREM OF (0, 1, 3) — INTERPOLATION

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1. The (0, 1, 3) — interpolation concerns the investigation of the polynomials $R_n(x)$ if they exist and are unique, of degree atmost $3n-1$ which take at the n given points $x_{\nu n}$ $(\nu=1, 2, \ldots, n)$ the arbitrary values $\alpha_{\nu n}$, whose first and third derivatives take at the same points the values $\beta_{\nu n}$ and $\gamma_{\nu n}$ respectively. We have solved in [4] the above interpolation problem for a particular choice of the points $x_{\nu n}$ which are the n real zeros of the polynomial $(1-x^2)P'_{n-1}(x)$ where $P_{n-1}(x)$ is the $(n-1)$th *Legendre polynomial.* For this choice of the abscissas we have shown in the first part of our work [4] that these polynomials exist and are unique only when n is even. In the other part of our work [5] we have studied the polynomials $R_n(x, f)$ for their convergence behavior when the numbers $\alpha_{\nu n}$ and $\beta_{\nu n}$ are taken to be the values of a function $f(x)$ and its first derivative respectively at the points $x_{\nu n}$. There we have shown that the sequence $R_n(x, f)$ converges uniformly to $f(x)$ in $[-1, 1]$ if $f(x)$ is continuously differentiable of order 2 in $[-1, 1]$.

In this paper we shall be concerned with the convergence (which we shall see, does not require the differentiability of the interpolatory function) of the polynomials $R_n(x, f)$ satisfying the following requirements:

$$\left.\begin{aligned} R_n(x_{\nu n}, f) &= f(x_{\nu n}) \\ R'_n(x_{\nu n}, f) &= 0 \\ R''_n(x_{\nu n}, f) &= \gamma_{\nu n} \end{aligned}\right\} \quad \nu = 1, 2, \ldots, n. \tag{1}$$

2. For the form of the polynomials $R_n(x, f)$ we obviously have [4]

$$R_n(x, f) = \sum_{\nu=1}^{n} f(x_{\nu n})\, u_{\nu n}(x) + \sum_{\nu=1}^{n} \gamma_{\nu n}\, w_{\nu n}(x) \tag{2}$$

where [4, § 11] $u_{\nu n}(x)$, $w_{\nu n}(x)$ are the uniquely determined polynomials of degree $\leqslant 3n-1$.

We shall prove the following

T h e o r e m. *Let the continuous function $f(x)$ satisfy the condition*

$$[f(x+h)-2f(x)+f(x-h)] = o(h) \quad (x-h, x+h) \in [-1, 1] \tag{3}$$

and the numbers $\gamma_{n\nu}$ of interpolatory polynomial (2) *satisfy the condition uniformly in* ν

$$\begin{aligned} &\gamma_{\nu n} = o(n^2)\,(1-x_{\nu n}^2)^{-1} \quad (\nu = 2, 3, \ldots, n-1) \\ &\gamma_{\nu n} = o(n^4), \qquad \gamma_{nn} = o(n^4). \end{aligned} \tag{4}$$

Then the sequence $R_n(x, f)$ converges uniformly to $f(x)$ in $[-1, 1]$.

3. The proof of this theorem mainly depends upon the following

Lemma. *Let the continuous function $f(x)$ $(-1 \leq x \leq 1)$ satisfy the condition* (3). *Then there is a polynomial $\Phi_n(x)$ of atmost degree n satisfying the following properties:*

(5) $$f(x)-\Phi_n(x)=o(n^{-1})(\sqrt{1-x^2}+n^{-1})$$

and

(6) $$\Phi_n'''(x)=o(n^2)\,\mathrm{Min}\,[(1-x)^{-1}, n^2]$$

uniformly in $x \in [-1, 1]$.

Proof. Part (5) of the above lemma is the particular case of the theorem of *G. Freud* [2] while (6) follows as a consequence of (5). We follow the same method of proof as given by *G. Freud* [3].

We define the numbers n_j $(j=0, 1, 2, \ldots, r)$ by

$$n_0=n,\ n_1=\left[\frac{n}{2}\right],\ \ldots,\ n_{j+1}=\left[\frac{n_j}{2}\right],\ \ldots,\ n_r=1;$$

$$r=\left[\frac{\log n}{\log 2}\right]+1.$$

We now have

(7) $$\Phi_n(x)=\sum_{j=0}^{r-1}[\Phi_{n_j}(x)-\Phi_{n_{j+1}}(x)]+\Phi_1(x).$$

From (5) we have

(8) $$\begin{aligned}\Phi_{n_j}(x)-\Phi_{n_{j+1}}(x)&=[\Phi_{n_j}(x)-f(x)]+[f(x)-\Phi_{n_{j+1}}(x)]\\&=o(n_j^{-1})(\sqrt{1-x^2}+n_j^{-1})\end{aligned}$$

which on using the inequality of *Dzyadyk* [1] gives:

(9) $$\begin{aligned}\Phi_{n_j}'''(x)-\Phi_{n_{j+1}}'''(x)&=o(n_j^2)\,\mathrm{Min}\,[(1-x^2)^{-1}, n_j^2]\\&=o\left(\frac{n^2}{2^{2j}}\right)\mathrm{Min}\,[(1-x^2)^{-1}, n^2].\end{aligned}$$

Hence from (7) and (9) we have

(10) $$\begin{aligned}\Phi_n'''(x)&=\sum_{j=0}^{r-1}[\Phi_{n_j}'''(x)-\Phi_{n_{j+1}}'''(x)]\\&=\sum_{j=0}^{r-1}o\left(\frac{n^2}{2^{2j}}\right)\mathrm{Min}\,[(1-x^2)^{-1}, n^2]\\&=o(n^2)\,\mathrm{Min}\,[(1-x^2)^{-1}, n^2].\end{aligned}$$

This completes the proof of our lemma.

4. We shall further need the following inequalities which have been proved in our work [5].

(11) $$w_{1n}(x)=O(n^{-5}),\quad w_{nn}(x)=O(n^{-5})$$

(12) $$w_{\nu n}(x)=O(n^{-\frac{7}{2}})\,l_{\nu n}(x)\,(1-x_{\nu n}^2)\,\nu+O(n^{-5})\,\nu^2,\quad 2\leq \nu \leq \frac{n}{2}$$

$$(13)\quad w_{\nu n}(x)=O(n^{-\frac{7}{2}})\,l_{\nu n}(x)\,(1-x_{\nu n}^2)(n-\nu)+O(n^{-5})(n-\nu)^2,\quad \frac{n}{2}+1\leqslant\nu\leqslant n-1$$

$$(14)\qquad u_{1n}(x)=O(n),\qquad u_{nn}(x)=O(n)$$

$$(15)\quad u_{\nu n}(x)=O(n^{\frac{1}{2}})\frac{l_{\nu n}(x)}{\nu}+O(1)\,\nu^{-\frac{5}{2}}+O(1)\,\nu^{\frac{1}{2}}\,l_{\nu n}^2(x)+O(1)\,l_{\nu n}^3(x),\quad 2\leqslant\nu\leqslant\frac{n}{2}$$

and

$$(16)\quad u_{\nu n}(x)=O(n^{\frac{1}{2}})\frac{l_{\nu n}(x)}{n-\nu}+O(1)(n-\nu)^{-\frac{5}{2}}+O(1)(n-\nu)^{\frac{1}{2}}\,l_{\nu n}^2(x)+O(1)\,l_{\nu n}^3(x),$$

$$\frac{n}{2}+1\leqslant\nu\leqslant n-1.$$

We shall also need the results [3]

$$(17)\qquad \frac{c_1}{n}\nu\leqslant(1-x_{\nu n}^2)^{\frac{1}{2}}\leqslant\frac{c_2}{n}\nu\qquad\left(2\leqslant\nu\leqslant\frac{n}{2}\right)$$

$$(18)\qquad \frac{c_1}{n}(n-\nu)\leqslant(1-x_{\nu n}^2)^{\frac{1}{2}}\leqslant\frac{c_2}{n}(n-\nu)\qquad\left(\frac{n}{2}+1\leqslant\nu\leqslant n-1\right)$$

where c_1 and c_2 are suitable numerical constants, and the following important result of *Fejér:*

$$(19)\qquad l_j^2(x)\leqslant\sum_{j=1}^{n}l_j^2(x)\leqslant 1\qquad(-1\leqslant x\leqslant 1;\ j=1,2,\ \dots\ ,n).$$

5. We now come to the proof of the theorem. According to the usual convention there holds

$$(20)\qquad R_n(x,f)-f(x)=R_n(x;f-\Phi_n)+\Phi_n(x)-f(x)$$

$$=\sum_{\nu=1}^{n}[f(x_{\nu n})-\Phi_n(x_{\nu n})]\,u_{\nu n}(x)+\sum_{\nu=1}^{n}[\gamma_{\nu n}-\Phi_n'''(x_{\nu n})]\,w_{\nu n}(x)+o(1).$$

From (5), (14), (15), (17) and (19) it follows that

$$\sum_{\nu=1}^{n}[f(x_{\nu n})-\Phi_n(x_{\nu n})]\,u_{\nu n}(x)=o(n^{-2})\,O(n)+\sum_{\nu=2}^{\frac{n}{2}}o(n^{-1})\frac{\nu}{n}\,O(n^{\frac{1}{2}})\frac{l_{\nu n}(x)}{\nu}$$

$$+\sum_{\nu=2}^{\frac{n}{2}}o(n^{-1})\frac{\nu}{n}\nu^{-\frac{5}{2}}+\sum_{\nu=2}^{\frac{n}{2}}o(n^{-1})\frac{\nu}{n}\cdot\nu^{\frac{1}{2}}\,l_{\nu n}^2(x)+\sum_{\nu=2}^{\frac{n}{2}}o(n^{-1})\frac{\nu}{n}\,l_{\nu n}^3(x)$$

$$(21)\qquad =o(n^{-1})+o(n^{-\frac{3}{2}})\sum_{\nu=2}^{\frac{n}{2}}l_{\nu n}(x)+o(n^{-2})\sum_{\nu=2}^{\frac{n}{2}}\nu^{-\frac{3}{2}}+o(n^{-2})\sum_{\nu=2}^{\frac{n}{2}}\nu^{\frac{3}{2}}\,l_{\nu n}^2(x)$$

$$+o(n^{-2})\sum_{\nu=2}^{\frac{n}{2}}\nu\,l_{\nu n}^3(x)=o(n^{-1})+o(n^{-2})\sum_{\nu=2}^{\frac{n}{2}}\nu^{\frac{3}{2}}\,l_{\nu n}(x)$$

$$=o(n^{-1})+o(n^{-2})\left(\sum_{\nu=2}^{\frac{n}{2}}\nu^3\right)^{\frac{1}{2}}\left(\sum_{\nu=2}^{\frac{n}{2}}l_{\nu n}^2(x)\right)^{\frac{1}{2}}=o(n^{-1})+o(1)=o(1).$$

Again from (4), (6), (11), (12), (17) and (19) we get

$$\sum_{\nu=1}^{\frac{n}{2}} [\gamma_{\nu n} - \Phi_n^{'''}(x_{\nu n})] \cdot w_{\nu n}(x)$$

$$= o(n^4)\, O(n^{-5}) + \sum_{\nu=2}^{\frac{n}{2}} o(n^2) \frac{n^2}{\nu^2} O(n^{-\frac{7}{2}})\, l_{\nu n}(x) \frac{\nu^2}{n^2} \cdot \nu + \sum_{\nu=2}^{\frac{n}{2}} o(n^2) \frac{n^2}{\nu^2} O(n^{-5})\, \nu^2$$

$$(22) \qquad = o(n^{-1}) + o(n^{-\frac{3}{2}}) \sum_{\nu=2}^{\frac{n}{2}} \nu\, l_{\nu n}(x) + o(1)$$

$$= o(n^{-1}) + o(n^{-\frac{3}{2}}) \Big(\sum_{\nu=2}^{\frac{n}{2}} \nu^2\Big)^{\frac{1}{2}} \Big(\sum_{\nu=2}^{\frac{n}{2}} l_{\nu n}^2(x)\Big)^{\frac{1}{2}} + o(1) = o(1).$$

In the same way we can prove

$$(23) \qquad \sum_{\nu=\frac{n}{2}+1}^{n} [f(x_{\nu n}) - \Phi_n(x_{\nu n})]\; u_{\nu n}(x) = o(1)$$

and

$$(24) \qquad \sum_{\nu=\frac{n}{2}+1}^{n} [\gamma_{\nu n} - \Phi_n^{'''}(x_{\nu n})]\, w_{\nu n}(x) = o(1).$$

Hence (20), (21), (22), (23) and (24) complete the proof of our theorem.

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