PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 1 (15), p. 109—110.

# A NOTE ON UNIFORM CONVERGENCE

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(Received 23. IV 1962)

**1.** Generalizing a classical theorem of C. Arzelá, P. S. Alexandroff has proved (see [1] or [2]), the following proposition:

*Let*

$$f_1, f_2, \ldots, f_n, \ldots$$

*be a convergent sequence of continuous mappings of a topological space X into a metric space Y, and let*

$$f(x) = \lim_n f_n(x).$$

*Then, f is continuous if and only if this sequence converges quasi-uniformly.*

The aim of this note is to show how this result can be extended from metric to uniform spaces.

**2.** Let

(1) $$\{f_\alpha; \quad \alpha \in D\}$$

be a convergent net of mappings of a topological space $X$ into a uniform space $(Y, \mathcal{U})$ and let

$$f = \lim_{\alpha \in D} f_\alpha .$$

The net (1) is said to converge *uniformly* to $f$ if and only if for each $U \in \mathcal{U}$, there is an $\alpha_U \in D$, such that

$$(f(x), f_\alpha(x)) \in U,$$

for each $\alpha \geqslant \alpha_U$ and each $x \in X$ ($\geqslant$ denotes the order relation in the directed set $D$).

We shall say that the net (1) *converges uniformly in the generalized sense* to $f$, provided $f = \lim\limits_\alpha f_\alpha$ and for each $U \in \mathcal{U}$ there is an $\alpha_U \in D$ such that

$$(f(x), f\alpha_U(x)) \in U,$$

for each $x \in X$.

It is obvious that the net

$$\{f\alpha_U; \quad U \in \mathcal{U}\}$$

converges uniformly to $f$.

The net (1) will be called *quasi-uniformly convergent* to $f$ provided $f = \lim\limits_\alpha f_\alpha$ and for each $U \in \mathcal{U}$ and $x_0 \in X$ there is a neighborhood $O_U$ of $x_0$ and an element $\alpha_U \in D$ such that

$$(f(x), f\alpha_U(x)) \in U,$$

for each $x \in O_U$.

Note that the quasi-uniform convergence can be interpreted as local uniform convergence in the generalized sense.

Now we can prove this

Theorem. *The convergent net* $\{f\alpha;\ \alpha \varepsilon D\}$ *of continuous mappings of a topological space* $X$ *into a uniform space* $(Y, \mathcal{U})$ *is convergent to a continuous mapping*

$$f = \lim_{\alpha} f_\alpha$$

*if and only if it converges quasi-uniformly.*

*Proof.* First, suppose the net (1) converges to $f$ quasi-uniformly and let $x_0 \varepsilon X$. For the neighborhood $U[f(x_0)]$, $U \varepsilon \mathcal{U}$, of the point $f(x_0)$, choose a symmetric $U \varepsilon \mathcal{U}$ such that

$$V \circ V \circ V \subset U.$$

Let $\alpha_V \varepsilon D$ and the neighborhood $O_V$ of $x_0$ be chosen in such a manner that

(2) $$(f(x), f\alpha_V(x)) \varepsilon V, \text{ for all } x \varepsilon O_V.$$

Since $f_{\alpha\ V}$ is continuous, there exists a neighborhood $O'$ of $x_0$, $O' \subset O_V$ satisfying

$$f_{\alpha\ V}[O'] \subset V[f_{\alpha\ V}(x_0)]$$

or

(3) $$(f\alpha_V(x_0), f\alpha_V(x)) \varepsilon V, x \varepsilon O'.$$

For $x = x_0$, (2) yields

(4) $$(f(x_0), f\alpha_V(x_0)) \varepsilon V.$$

Finally (2), (3) and (4) imply

$$(f(x_0), f(x)) \varepsilon V \circ V \circ V \subset U,$$

i. e.

$$f(x) \varepsilon U[f(x_0)], \text{ for all } x \varepsilon O'.$$

Hence, the function $f$ is continuous at $x_0$.

Conversely, let $f$ be continuous. Without loss of generality we can suppose $U \varepsilon \mathcal{U}$ open in the product topology of $Y x Y$. Let

$$\Gamma_\alpha = \{x : x \varepsilon X,\ (f(x), f_\alpha(x)) \varepsilon U\}.$$

It is easy to see that

$$X = \bigcup_{\alpha \varepsilon D} \Gamma_\alpha$$

and all $\Gamma_\alpha$ are open. Hence, for each $x_0 \varepsilon X$, there is a neighborhood $\Gamma_{\alpha_0}$ of $x_0$, such that

$$(f(x), f_{\alpha_0}(x)) \varepsilon U, \text{ for all } x \varepsilon \Gamma_{\alpha_0},$$

thus proving that $f_\alpha$ converge to $f$ quasi-uniformly.

## REFERENCES

[1] П. С. Александров: *О так называемой квазиравномерной сходимости*, УМН, том III, выпуск I (23), 1948, стр. 213—215.

[2] П. С. Александров: *Введение в общую теорию множеств и функций*, Москва 1948.