PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 1 (15), p. 5—10.

ON ROOTS OF AN ELEMENT OF A BANACH ALGEBRA[1*]

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(Received 30. III 1961)

An element b of a *Banach algebra* B is called an n-th root of $a \in B$ if

$$b^n = a$$

holds. It is well known (see Theorem 1 below) that an element $a \in B$ need not possess any root in B. If B is an algebra of matrices of finite order and if $a \in B$ is regulai, then it possesses any root in B. It is an interesting result of [1] and [2] that in the case in which B is a *Banach algebra* of bounded operators on a *Hilbert* space a regularity of an element $a \in B$ does not imply the existance of a square (n-th) root of a. *Moreover* as it is proved in [2] the set of elements in B which possess square roots is not dense in B.

It is the object of this paper to prove (Theorem 2) that any *Banach algebra* B can be imbeded isomorphycally and isometrically in another *Banach algebra* B' in such a way, that any element of B as an element of B' possesses any root in B'. This is generalisation of our result obtained in [4], in which B was imbeded in B' isomorphically, but not isometrically in such a way that any element of B possessed a square root in B'. Furtheremore (Theorem 3) the algebra B can be isomorphycally and isometrically imbeded in a normed algebra B'' which has the property that any element of B'' possesses any root in B''.

Beside this in Theorem 1 we find necessary and sufficient conditions for an entire function f in order that f maps a *Banach algebra* B of matrices of (any) finite order on B.

Theorem 1. *Let*

$$f(z) = \sum_0^\infty \alpha_n z^n$$

be an entire function and $B = \{J, S, T, W, \ldots\}$ *a Banach algebra of all matrices, with complex matrix elements, of finite (any) order.*

Then, the mapping

$$T \to f(T) = \sum_0^\infty \alpha_n T^n$$

of B *in* B *is on* B *if and only if for any complex number* α *the set*

$$\Gamma_\alpha = \{z \mid f(z) = \alpha, \quad f'(z) \neq 0\}$$

is not empty.

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[1] The author wishes to thank Professor *John M. Horwath* and Professor *James A Hummel* for their interest and fruitfull discussion during the preparation of this paper.

* This work was supported by the National Science Fundation.

Proof: Necessity. Suppose that Γ_α is empty for some α_0. Then at least one of the following two cases occur: 1. $f(z)=\alpha_0$ has no solution or 2. $f'(z)=0$ whenever $f(z)=\alpha_0$. In the first case f is not a mapping onto already in the case B is the algebra of complex numbers. In the second case for B take the algebra of all 2×2 matrices and consider the equation:

$$f(T)=\alpha_0\, E+J$$

where E is the unit matrix, $J_{21}=1$ and all other matrix elements of J are zero. Since T commutes with $f(T)$ it also commutes with J and therefore

$$T=\alpha E+\alpha_1 J.$$

Since α is a single eigenvalue of T we have $f(\alpha)=\alpha_0$. Now $\alpha_1=0$ would imply $f(T)=\alpha_0 E$ which is impossible. Therefore $\alpha_1\neq 0$. But $\alpha_1\neq 0$ implies the existence of a regular matrix S such that

$$S^{-1}TS=\alpha E+J$$

from which follows:

$$S^{-1}f(T)\,S=f(\alpha E+J)=f(\alpha)E+f'(\alpha)J=\alpha_0 E$$

because $f'(\alpha)=0$. Hence again $f(T)=\alpha_0 E$ which is impossible.

Sufficiency. Suppose that B is an algebra of all $n\times n$ matrices with complex elements and let V be any element of B. We assert that the equation

$$f(T)=V \tag{1}$$

has at least one solution T in B. Let S be a regular matrix with a property that

$$U=S^{-1}VS$$

is a *Jordan* form of V, i. e.

$$U=\Sigma\dot{+}(\alpha_k E_k+J_k) \tag{2}$$

where E_k is a unit matrix of the order n_k; $J_k=0$ if $n_k=1$ and for $n_k>1$ $(J_k)_{21}=\cdots=(J_k)_{n_k,\,n_k-1}=1$ and all other matrix elements of J_k are zero. The symbol $\dot{+}$ denotes the direct sum of matrices. Now, consider the equation

$$f(W_k)=\alpha_k E_k+J_k \tag{3}$$

where W_k is a matrix of the order n_k. By the assumption about the function f there exists at least one number z_k such that $f(z_k)=\alpha_k$ and $f'(z_k)\neq 0$. Now,

$$f(z_k E_k+J_k)=f(z_k)E_k+\frac{f'(z_k)}{1!}J_k+\cdots+\frac{f^{(n-1)}(z_k)}{(n_k-1)!}J^{n_k-1}$$

and $f'(z_k)\neq 0$, $f(z_k)=\alpha_k$ imply the existence of a regular matrix S_k of the order n_k such that

$$S_k^{-1}f(z_k E_k+J_k)\,S_k=\alpha_k E_k+J_k.$$

Thus the matrix:

$$W_k=S_k^{-1}(z_k E_k+J_k)\,S_k$$

satisfies (3). But then the matrix

$$W=\Sigma\dot{+}W_k$$

has the property that $f(W)=U$. Thus the matrix

$$T=SWS^{-1}$$

satisfies (1).

Observe that functions $z^n (n > 1)$, $\sin z$, $\cos z$, $\exp z$ do not satisfy the conditions of Theorem 1. Thus matrix equations $T^n = V (n > 1)$ $\sin T = V$, $\cos T = V$, $\exp T = V$ have not always solutions. On the other hand functions

$$f(z) = z^3 - z, \; f(z) = \int_0^z \exp [\exp t] \, dt$$

satisfy all conditions of theorem 1 ([5], p. 257).

From the proof of Theorem 1 we see that if f is an entire function and if $f(T) = V$ has always a solution in the *Banach algebra* of matrices of second order then the function f satisfies all conditions of Theorem 1.

Theorem 2. *Let $\Phi = \{\alpha, \beta, \dots\}$ be a field of real or complex numbers and $B = \{a, b, \dots\}$ a Banach algebra over the field Φ with a unit e.*

Then, there exists a Banach algebra $B' = \{T, S, \dots\}$ over Φ with a unit E and a mapping (imbedding) $\theta : B \to B'$ such that

1) $\theta (\alpha a + \beta b) = \alpha \theta (a) + \beta \theta (b) \qquad (\alpha, \beta \in \Phi, \; a, b \in B)$

2) $\theta (ab) = \theta (a) \, \theta (b)$

3) $\| \theta (a) \| = \| a \|$ *and*

4) $\theta (a)$ *and a have the same spectrum.*

Furthermore, if a is any element of B and n any natural number then, there exists at least one element $T \in B'$ such that:

I. $T^n = \theta (a)$ *and*

II. *If $b \in B$ commutes with a, then T commutes with $\theta (b)$.*

Proof: Consider the set of all sequences:

$$x = (x_1, x_2, \dots), \quad x_j \in B$$

for which

$$\sum_0^\infty \| x_j \| < + \infty .$$

This set, which we denote by X, is a *Banach space* over Φ with the usual definitions of the addition and multiplication with a number from Φ and with the norm:

$$\| x \| = \sum_0^\infty \| x_j \| .$$

Let B' be the set of all bounded and linear operators T which are defined on X and have ranges in X. The set B' becomes a *Banach algebra* in the usual way by the norm:

$$\| T \| = \sup \| Tx \| \; (x \in X, \; \| x \| \leqslant 1, \quad T \in B').$$

Now, if a_{ij} $(i, j = 1, 2, \dots)$ are elements of B such that

$$(4) \qquad \begin{cases} \sum_{i=1}^\infty \| a_{ij} \| \leqslant M < + \infty & j = 1, 2, \dots \text{ and} \\ \sum_{j=1}^\infty \| a_{ij} \| \leqslant M & i = 1, 2, \dots \end{cases}$$

where M is a constant, then $y=Tx$ defined by

$$(5)\qquad y_i=\sum_{j=1}^{\infty} a_{ij}\, x_j, \qquad \sum_{j=1}^{\infty} \| x_j \| < +\infty$$

is a bounden and linear operator on X. Indeed, the series (5) converges in B for any $i=1, 2, \ldots$ and

$$\sum_{i=0}^{\infty} \| y_i \| \leqslant \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \| a_{ij} \| \; \| x_j \| \leqslant \sum_{j=1}^{\infty} \| x_j \| \left(\sum_{i=1}^{\infty} \| a_{ij} \| \right) \leqslant M \cdot \sum_{j=1}^{\infty} \| x_{ij} \|, \text{ i. e.}$$

$$\| Tx \| \leqslant M \| x \|$$

for any $x \in X$. For any $a \in B$ the operator aE defined by the matrix $a\,\delta_{ij}$ ($\delta_{ij}=e$ if $i=j$ and the null-element of B otherwise) is an element of B'.

We set:

$$(6)\qquad \theta(a)=aE.$$

Obvously θ satisfies the conditions $1-4$ of Theorem 2. It remains to prove that every element aE possesses n-th root in B'.

Suppose that $a \in B$ and a natural number n are given. Consider an $n \times n$ matrix T_n, matrix elements of which belong to B[2]), such that:

$$[T_n]_{1n}=a, \; [T_n]_{21}=\cdots=[T_n]_{n,\,n-1}=e$$

and all other matrix elements of T_n are equal to the null-element of B. Then:

$$[(T_n)^n]_{ij}=\sum_{k=1}^{n} (T_n)_{ik}\, [(T_n)^{n-1}]_{kj}=[(T_n)^{n-1}]_{i-1,\,j}=\cdots=[(T_n)^{n-i+1}]_{1,\,j}$$

$$=\sum_{k=1}^{n} (T_n)_{ik}\, [(T_n)^{n-i}]_{kj}=a\,[(T_n)^{n-i}]_{nj}=a\sum_{k=1}^{n} (T_n)_{nk}\, [(T_n)^{n-i-1}]_{kj}$$

$$=a\,[(T_n)^{n-i-1}]_{n-1,\,j}=\cdots=a\,(T_n)_{i+1,\,j}.$$

Thus

$$[T_n]^n=aE_n,$$

where E_n is an $n \times n$ diagonal matrix with e on the main diagonal. Now, the matrix elements of the matrix

$$T=\sum \dot{+}\, T_n=\left\| \begin{matrix} \boxed{T_n} & & & \\ & \boxed{T_n} & & \\ & & \boxed{} & \\ & & & \ddots \end{matrix} \right\|$$

satisfy conditions (4) with $M=1+\|a\|$ and therefore (5) defines an element T of B'. Obviously

$$T^n=\sum \dot{+}\, [T_n]^n=\sum \dot{+}\, aE_n=a\sum \dot{+}\, E_n=aE=\theta(a).$$

Furhermore if $ab=ba$ $(a, b \in B)$, then T and a „scalar" matrix $\theta(b)$ commute. Thus Theorem 2 is proved.

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[2] For general definition of a matrix see [3].

Notice that two elements a and b of B may commute, but their constructed roots need not commute.

Theorem 3. *Let B be a Banach algebra over a field Φ of complex or real numbers with a unit e.*

There exists a n o r m e d *algebra B' such that for any S of B' and any natural number n there exists at least one element $T \in B'$ such that*

$$T^n = S.$$

The Banach algebra B can be isomorphycally and isometrically imbeded in B'.

Proof: Let ω denote the ordinal number of the set of all natural numberes and denote $\Omega = \omega^\omega$.

The set of all sequences

$$x = (x_0, x_1, \dots, x_\alpha, \dots)_{0 \leqslant \alpha < \Omega}$$

$x_\alpha \in B$ for which

$$\|x\| = \sum_{0 \leqslant \alpha < \Omega} \|x_\alpha\| < +\infty$$

is a *Banach space* if the addition of two elements and the multiplication with and element of Φ is defined in the usual way. For an element $a \in B$ and $x \in X$ define:

$$ax = (ax_0, ax_1, \dots, ax_\alpha, \dots)_{0 \leqslant \alpha < \Omega} \quad \text{and}$$

$$xa = (x_0 a, x_1 a, \dots, x_\alpha a, \dots)_{0 \leqslant \alpha < \Omega}.$$

Obviously ax and xa are in X. By f_α, $0 \leqslant \alpha < \Omega$, denote a vector which has the α-th coordinate e-the unit of B, and other coordinates the null-element of B. Then:

$$x = \sum_{0 \leqslant \alpha < \Omega} x_\alpha f_\alpha = \sum_{0 \leqslant \alpha < \Omega} f_\alpha x_\alpha .$$

By a bounded linear operator T: $X \to X$ we mean a mapping which has the following properties:

1) $T(\lambda x + \mu y) = \lambda Tx + \mu Ty$ for all $x, y \in X$ and $\lambda, \mu \in \Phi$.

2) $\|T\| = \sup \|Tx\| \quad (x \in X, \|x\| \leqslant 1)$ is finite and

3) $T(xa) = (Tx)a$ for any $a \in B$ and $x \in X$.

The set of all bounded linear operators form a *Banach algebra* B'' over Φ.

If $T \in B''$ and $x \in X$, then

$$Tx = T \sum_{0 \leqslant \alpha < \Omega} f_\alpha x_\alpha = \sum_{0 \leqslant \alpha < \Omega} (Tf_\alpha) x_\alpha .$$

Hence T is completely determined by its values on the basic set $\{f_\alpha\}$. Setting

$$Tf_\alpha = \sum_{0 \leqslant \beta < \Omega} f_\beta T_{\beta\alpha}$$

with $T_{\beta\alpha} \in B$ we find that to the product $T' T''$ of two elements of B'' there corresponds the matrix:

$$(T' T'')_{\alpha\beta} = \sum_{0 \leqslant \gamma < \Omega} (T')_{\alpha\gamma} (T'')_{\gamma\beta}$$

and also to the sum of operators there corresponds the sum of the corresponding matrices. In such a way we have an isomorphism of B'' with a set of bounded matrices, provided that a bounded matrix is defined in usual way, and matrices in question are matrices of the order Ω with matrix elements in B.

A matrix $T=(T_{\alpha\beta})$ such that for some $0 \leqslant \sigma < \Omega$

$$T_{\alpha\beta} = T_{\sigma\cdot\gamma+\alpha,\, \sigma\cdot\gamma+\beta}$$

for all $0 \leqslant \alpha, \beta, \gamma < \Omega$, and all other matrix elements of T vanish will be called a direct sum of a matrix

$$(T_{\alpha\beta})_{0\leqslant\alpha,\,\beta<\sigma}$$

which is of the order σ. We will write:

$$T = \sum \dot{+} [(T_{\alpha\beta})_{0\leqslant\alpha,\,\beta<\sigma}]$$

If X_σ is a subspace of X determined by the vectores $f_0, \ldots, f_\alpha, \ldots, \alpha<\sigma$, then the set of all bounded linear operators from X_σ in X_σ form a *Banach algebra* $B(\sigma)$. To any element of $B(\sigma)$ there corresponds a bounded matrix

$$[T_{\alpha\beta}]_{0\leqslant\alpha,\,\beta<\sigma}$$

of the order σ. Taking a direct sum of such a matrix we get a matrix of the order Ω which represents an element of B''. By this construction, from $B(\sigma)$ we get a *Banach algebra* $B'(\sigma)$ which is a subalgebra of B''.

Set $B_n = B'(\omega^n)$ and $\omega^0 = 1$. Then the algebra B_0 consists of „scalar" matrices only. Thus, an element $a \in B$ can be identified with the corresponding matrix which is in B_0 and which is obtained as a direct sum of the matrix a of the order $\delta = 1$. Furthermore we have:

$$B_0 \subset B_1 \subset \cdots \subset B_n \subset \cdots \subset B''.$$

The cannonical mapping of B_n into B_{n+1} or B'' is an isometric and isomorphyc imbeding of B_n into B_{n+1} or B'' respectively. Suppose tat a natural number $m>1$ is given and that S is an element of B_n. We assert, that there exists at least one element $T \in B_{n+1}$ such that $T^m = S$. Indeed, any element of $B(\omega^{n+1})$ can be considered as a block-matrix of order ω with block-matrices of order ω^n. Thus, elements of $B(\omega^n)$ appear as „scalar" matrices in $B(\omega^{n+1})$. On the other hand S is a direct sum of an element $S_0 \in B(\omega^n)$. The element S_0 is a „scalar" matrix in $B(\omega^{n+1})$ and we can, as in Theorem 2, in terms of S_0 define $T_0 \in B(\omega^{n+1})$ in such a way, that $T_0^m = S_0$. However, here we also take into account the obvious fact, that the multiplication of block-matrices satisfies the usual rules of multiplication of block-matrices with complex matrix elements. If T is a direct sum of T_0, then T is in B_{n+1} and obviously $T^m = S$. Thus any element of B_n as an element of B_{n+1} possesses any root in B_{n+1}. Set

$$B' = \bigcup_{0\leqslant n<\omega} B_n.$$

Plainly B' is a subalgebra of B''. It is a normed algebra and any element of B' possesses any root in B'.

Q. E. D.

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