THE AREA OF A GENERALIZED CIRCLE IN THE HYPERBOLIC PLANE

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If M is a bounded and closed set of the hyperbolic or Euclidean plane and $\tilde{M}$ the convex hull of M, then the area μM_r of the circle M_r around M with radius r and the area $\mu \tilde{M}_r$ of the circle $\tilde{M}_r$ around $\tilde{M}$ have the property that

$$\lim_{r \to \infty} (\mu \tilde{M}_r - \mu M_r)$$

exists. If M is in the Euclidean plane then this limit is zero, but if M is in the hyperbolic plane and if it does not contain the boundary of $\tilde{M}$ then this limit is $+\infty$.

If M consists of two points of an n-dimensional Euclidean space $(n \geqslant 2)$, $\tilde{M}$ is the convex hull of M and M_r resp. $\tilde{M}_r$ the spheres of radius r around M resp. $\tilde{M}$ then

$$\mu \tilde{M}_r - \mu M_r = O(r^{n-3})$$

where μS denotes the n-dimensional measure (the volume) of S.

Furthermore, for a convex set M in the hyperbolic plane (1) and (2) are valid, where $\nu F(M)$ is the length of the boundary $F(M)$ of M. These results are formulated in the following theorems:

THEOREM 1. *Let X be the hyperbolic plane, $d(x, y)$ the distance between two points $x, y \in X$, μS the area (two-dimensional Lebesgue's measure) of $S \subseteq X$, $F(S)$ the boundary of S and $\nu F(S)$ the length (one-dimensional Lebesgue's measure) of $F(S)$.*

If $M \subset X$ is a non-linear convex, closed and bounded set and

$$M_r = \{x \mid d(x, M) \leqslant r, x \in X, r > 0\}$$

a generalized circle around M with radius r, then

$$\mu M_r = \mu M \operatorname{ch} r + \nu F(M) \operatorname{sh} r + 2\pi(\operatorname{ch} r - 1) \tag{1}$$

and

$$\nu F(M_r) = \frac{d}{dr}(\mu M_r). \tag{2}$$

THEOREM 2. *Let X be a Euclidean plane, $d(x,y)$, μS, $\nu F(S)$ the corresponding functions for the Euclidean plane.*

If M is a closed and bounded set and $\tilde{M}$ the convex hull of M (i. e. the smallest convex set which contains M) then

$$\lim_{r\to\infty} (\mu \tilde{M}_r - \mu M_r) = 0.$$

In the case when M is a continuum of the Euclidean plane, theorem 2 was proved by G. Fast ([1], theorem 6, p. 162) who has studied more carefully μM_r as a function of r.

THEOREM 3. *Let X be a hyperbolic plane, $M \subset X$ a closed and bounded set, $\tilde{M}$ the convex hull of M, $F(\tilde{M})$ the boundary of $\tilde{M}$ in the case $\mu \tilde{M} > 0$ and $F(\tilde{M}) = \tilde{M}$ otherwise.*

If $M \supseteq F(\tilde{M})$ then $\mu \tilde{M}_r - \mu M_r = 0$ for every $r > \sup d(x,y)\,(x, y \in M)$ and if M does not contain $F(\tilde{M})$ then

$$\lim (\mu \tilde{M}_r - \mu M_r) = +\infty.$$

Proof of theorem 1. First, suppose that $M = P$ is a convex polygon with verteces $p_1, p_2, \ldots, p_n$, angles α_k at p_k and sides $a_k = \overline{p_k p_{k+1}}$ $(p_{n+1} = p_1)$. In p_k we construct the perpendiculars on a_{k-1} and a_k and denote the angle between them by α'_k. We have $\alpha_k + \alpha'_k = \pi$ and therefore

$$\sum_{k=1}^{n} (\alpha_k + \alpha'_k) = n\pi.$$

If

$$D(P) = (n-2)\pi - \sum_{k=1}^{n} \alpha_k$$

denotes the defect of P, then we find :

$$\sum_{k=1}^{n} \alpha'_k = 2\pi + D(P).$$

The boundary of P_r consists of arcs of circles of radius r corresponding to angles α'_k and of equidistant-curves with segments a_k on their axes and having the distance r. Thus we have:

$$\text{(3)} \qquad \begin{cases} \mu P_r = \mu P + \sum_{k=1}^{n} a_k \operatorname{sh} r + [2\pi + D(P)](\operatorname{ch} r - 1) \\ \nu F(P_r) = \sum_{k=1}^{n} a_k \operatorname{ch} r + [2\pi + D(P)] \operatorname{sh} r, \end{cases}$$

because the length of an arc of the equidistant-curve is $a \operatorname{ch} r$, and to an angle α on the circle corresponds the arc $\alpha \operatorname{sh} r$, while the corresponding areas are $a \operatorname{sh} r$ and $\alpha(\operatorname{ch} r - 1)$ respectively.

On the other hand numerically $D(P) = \mu P$. This and (3) imply:

$$\text{(4)} \qquad \begin{cases} \mu P_r = \mu P \operatorname{ch} r + \nu F(P) \operatorname{sh} r + 2\pi(\operatorname{ch} r - 1) \\ \nu F(P_r) = \mu P \operatorname{sh} r + \nu F(P) \operatorname{ch} r + 2\pi \operatorname{sh} r. \end{cases}$$

If M is a convex set then we can take a sequence $P^{(n)}$ of convex polygons which tends to M. Since the area and the length are continuous functions (4) implies:

$$\mu M_r = \mu M \operatorname{ch} r + \nu F(M) \operatorname{sh} r + 2\pi(\operatorname{ch} r - 1)$$
$$\nu F(M_r) = \mu M \operatorname{sh} r + \nu F(M) \operatorname{ch} r + 2\pi \operatorname{sh} r$$

from which we see that (1) and (2) hold true. Q.E.D.

For the proofs of theorems 2 and 3 we need the following well-known and obvious lemma 1.

LEMMA 1. *Let M be a closed and bounded set in the Euclidean plane or in the hyperbolic plane, $\tilde{M}$ the convex hull of M and $F(\tilde{M})$ the boundary of $\tilde{M}$.*

If $x \in F(\tilde{M})$ and $x \notin M$ then x is an interior point of an open interval which is disjoint with M but which is contained in $F(\tilde{M})$ and the end-points of which are in M.

Proof: For the element $x \in F(\tilde{M})$ there is at least one line p which passes through x and such that $\tilde{M}$ is on one side of p, say in the lower half-plane. We assert that p contains at least one point of the set M. Otherwise the distance of these two closed sets would be positive, i. e. $d(p, M) > 0$. Since $\tilde{M}$ is bounded, there is a point x' on p such that $\tilde{M}$ is on the right side of the perpendicular n on p at x'. On the line n in the lower half-plane construct a point x'' such that $0 < d(x', x'') < d(p, M)$ and through x'' draw the line q parallel with p on the side in which is $\tilde{M}$. Obviously M is in the lower half-plane with respect to q, $d(x, q) > 0$ and x is in the upper half-plane with respect to q. The intersection of the lower half-plane with respect to q with $\tilde{M}$ is a convex set which contains M and which does not contain $\tilde{M}$. Since this contradicts the definition of $\tilde{M}$ we conclude that the set

$$M_0 = M \cap p$$

is not empty. Now, $\tilde{M}_0$ belongs to $F(\tilde{M})$. We assert that x is an interior point of $\tilde{M}_0$. Suppose that this is not so and that $\tilde{M}_0$ is on the right side of x. Since M is closed and has no point on p which is on the left of x there is a line $p' \neq p$ through x such that M is on the right half-plane with respect to p'. Thus M is in the intersection of two half-planes determined by p and p'. Further $x \notin M$ implies $d(x, M) > 0$, i.e. there is a circle $\{x\}_r$ in which M has no points. This circle meets p' in the lower half-plane in y and p in the right half-plane in z. The line p'' through y and z divides the plane in two half-planes: in one is M and in the other is x. Since $d(x, p'') > 0$ there is a contradiction with the definition of $\tilde{M}$. Thus $\tilde{M}_0$ is not on the right side of x. On account of the same reason it is not on the left of x. Thus $x \in \tilde{M}_0$. Since $x \notin M$ we find that x is an interior point of the segment $\tilde{M}_0$, and therefore an interior point of the open interval which does not contain points of M_0 but end-points of which obviously belong to M. Q. E. D.

P r o o f of theorem 2. According to lemma 1 $F(\tilde{M})$ consists of points of M and of some open, with M disjoint, intervals with length a_k which have end-points in M.

Let $N = M \cap F(\tilde{M})$.

Since $N \subseteq M \subseteq \tilde{M}$ we have

$$N_r \subseteq M_r \subseteq \tilde{M}_r$$

and therefore

$$\mu N_r \leqslant \mu M_r \leqslant \mu \tilde{M}_r.$$

If

$$r > \sup_{x, y \in M} d(x, y), \quad a = \sup_k a_k,$$

then

$$0 \leqslant \mu\tilde{M}_r - \mu M_r \leqslant \mu\tilde{M}_r - \mu N_r \leqslant \sum_k (a_k r - a_k \sqrt{r^2 - a_k^2}) \leqslant$$

$$\leqslant \sum_k (a_k r - a_k \sqrt{r^2 - a^2}) = \left(\sum_k a_k\right)(r - \sqrt{r^2 - a^2}) \leqslant$$

$$\leqslant \nu F(\tilde{M}) \frac{a^2}{r + \sqrt{r^2 - a^2}}.$$

Since $\nu F(\tilde{M})$ is finite we deduce

$$\lim_{r \to \infty} (\mu\tilde{M}_r - \mu M_r) = 0.$$

Q. E. D.

COROLLARY 1. *If M is a bounded and closed set in the Euclidean plane and*

$$\mu M_r = \alpha + \beta r + \gamma r^2$$

for all r, then M is a convex set.

Proof: For a convex set M in the Euclidean plane the same arguments as in theorem 1 lead to the well-known formula due to Minkowski:

$$\mu M_r = \mu M + \nu F(M) r + r^2 \pi.$$

Thus:

$$\mu\tilde{M}_r = \mu\tilde{M} + \nu F(\tilde{M}) r + r^2 \pi$$

holds for all r. According to theorem 2 $\mu\tilde{M}_r - \mu M_r \to 0$ as $r \to \infty$. This can be only if $\alpha = \mu\tilde{M}$, $\beta = \nu F(\tilde{M})$ and $\gamma = \pi$, i.e.

$$\mu M_r = \mu\tilde{M} + \nu F(\tilde{M}) r + r^2 \pi. \tag{5}$$

If we pass to the limit $r \to 0$ in (5) we find

$$\mu M = \mu\tilde{M}. \tag{6}$$

If $\mu\tilde{M} = 0$, then $\tilde{M}$ is either a segment or a point. If $\tilde{M}$ is a point then M is also a point. If $\tilde{M}$ is a segment then one easily concludes that $\tilde{M} = M$. Suppose that $\mu\tilde{M} \neq 0$. If M is not a convex set, then two points $x_1, x_2 \in M$ exist such that the segment $\overline{x_1 x_2}$ is not in M. There is therefore a point

$x \in \overline{x_1 x_2}$ which is not in M. But then $d(x,M) > 0$, i.e. there is a circle $\{x\}_r$ such that:

$$\{x\}_r \cap M = \varnothing .$$

Since $\mu\tilde{M} > 0$, the set M possesses a point x_3 which is not on the same line as x_1, x_2. The triangle $\Delta = \Delta x_1 x_2 x_3$ is in $\tilde{M}$ and therefore:

$$\Delta \cap \{x\}_r \subseteq \tilde{M}$$

has positive measure. We have:

$$\mu(M \cap \Delta) < \mu(\tilde{M} \cap \Delta) = \mu\Delta. \tag{7}$$

Furthermore $M \cap C\Delta \subseteq \tilde{M} \cap C\Delta$ where $C\Delta$ denotes the complement of Δ. Thus,

$$\mu(M \cap C\Delta) \leqslant \mu(\tilde{M} \cap C\Delta). \tag{8}$$

If we add (7) and (8) we get:

$$\mu M < \mu\tilde{M}$$

which contradicts $\mu M = \mu\tilde{M}$. In such a way we have $M = \tilde{M}$, i.e. M is a convex set.

In the case when M is a continuum corollary 1 was proved by G. Fast ([1] theorem 5, p. 161) in a somewhat different way.

Proof of theorem 3. Since the first assertion is obvious we prove only the second assertion.

I. *The set M is not linear.* Since M does not contain $F(\tilde{M})$ there is a point $x \in F(\tilde{M})$ which is not in M. But then x is an interior point of a segment yz $(y,z \in M)$ which belongs to $F(\tilde{M})$ (lemma 1). Now $x \notin M$ implies $d(x,M) > 0$, i. e. there is a circle with radius $\varepsilon > 0$ such that

$$\{x\}_\varepsilon \cap M = \varnothing.$$

Obviously we can find a such that:

1. $a + 2\Pi(a/2) - \pi \neq 0$* where $\Pi(a/2)$ is the angle of parallelism of $a/2$ and

2. The Saccheri quadrilateral $x'y'z'x''$ (see the figure) is in the set

$$\{x\}_\varepsilon \cap \tilde{M},$$

its height is a and $\overline{xx'} = \overline{xx''} = a/2$.

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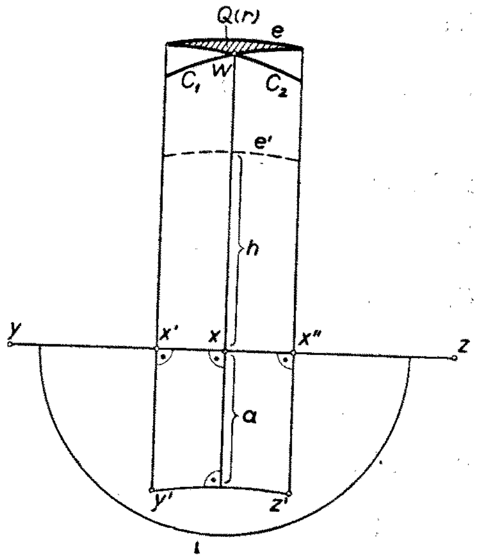
* It is easy to prove that $a + 2\Pi(a/2) - \pi > 0$ for $a > 0$.

Let M' be a closed set which is obtained by removing from $\tilde{M}$ the Saccheri quadrilateral $x'y'z'x''$. Obviously $M \subset M' \subset \tilde{M}$ and therefore

(9) $$M_r \subseteq M'_r \subseteq \tilde{M}_r$$

On the opposite side of M with respect to $x'x''$ construct two equidistant-curves e and e' with distance $r > \sup d(p,q)$ $(p, q \in M)$ and $x'x''$ resp. $y'z'$ as their axes, and two circles C_1, C_2 with radius r and with centers x'' resp. x'. Let w be the meeting point of arcs of these two circles which is on the same side as e with respect to $x'x''$. With H denote xw, put $h = r - a$ and by $Q(r)$ denote the set which is bounded by C_1, C_2 and e. From the triangle $x'xw$ we have $\operatorname{ch} r = \operatorname{ch}\frac{a}{2}\operatorname{ch} H$. Since $r = h + a > h + \frac{a}{2}$ we have:



$$\operatorname{ch}\frac{a}{2}\operatorname{ch} H > \operatorname{ch}\left(\frac{a}{2} + h\right) = \operatorname{ch}\frac{a}{2}\operatorname{ch} h + \operatorname{sh}\frac{a}{2}\operatorname{sh} h,$$

i. e.

$$\operatorname{ch}\frac{a}{2}(\operatorname{ch} H - \operatorname{ch} h) > 0$$

which implies $\operatorname{ch} H - \operatorname{ch} h > 0$, i.e. $H > h$. Thus the equidistant-curve e' has no common points with $Q(r)$. This implies:

$$Q(r) \cap \{\overline{yz}\}_r = \emptyset.$$

Let u be any point between x' and y'. We assert that:

(10) $$\{u\}_r \cap Q(r) = \emptyset.$$

Otherwise a point $v \in Q(r)$ would exist such that $\overline{uv} = r$. Consider the triangle, uvx'. The angle of this triangle at x' is $> \pi/2$. Thus the angle at

u is $<\pi/2$. However this implies $r=\overline{uv}>\overline{x'v}\geqslant r$ which is impossible. In the same way we see that (10) holds if $u\in M'$ is any point which is not between the lines $x'y'$ and $x''z'$. But if u is between y' and z' then as we have proved (10) holds too. This implies that (10) holds for every $u\in M'$ which is between these two lines, i.e. (10) holds for every $u\in M'$. Thus:

$$Q(r)\cap M'_r=\varnothing. \tag{11}$$

This and (9) imply:

$$Q(r)\subseteq \tilde{M}_r\setminus M'_r\subseteq \tilde{M}_r\setminus M_r$$

and therefore:

$$\mu Q(r)\leqslant\mu(\tilde{M}_r\setminus M'_r)\leqslant\mu(M_r\setminus M_r). \tag{12}$$

On the other hand

$$\mu Q(r)=a\operatorname{sh} r-[2\varphi(r)(\operatorname{ch} r-1)+\mu\Delta],$$

where Δ is the triangle $x'x''w$ and $\varphi(r)=\pi/2-\sphericalangle(xx'w)<\pi/2$. We can write $\mu Q(r)$ in the following form:

$$\mu Q(r)=2\varphi(r)-\mu\Delta-\frac{1}{2}[a+2\varphi(r)]\exp(-r)+\frac{1}{2}[a-2\varphi(r)]\exp r. \tag{13}$$

Now, $r\to\infty$ implies:

$$\mu\Delta\to\pi-2\Pi(a/2),\quad \varphi(r)\to\pi/2-\Pi(a/2)$$

and

$$a-2\varphi(r)\to a+2\Pi\left(\frac{a}{2}\right)-\pi\neq 0.$$

This and (13) implies

$$\lim_{r\to\infty}\mu Q(r)=+\infty \tag{14}$$

which together with (11) and (12) leads to:

$$\lim_{r\to\infty}\mu(\tilde{M}_r\setminus M_r)=\lim_{r\to\infty}(\mu\tilde{M}_r-\mu M_r)\geqslant\lim_{r\to\infty}Q(r)=+\infty.$$

II. *The set M is linear.* In this case $\tilde{M}$ is a segment. Since $M\neq\tilde{M}$ there is a segment $\overline{x'x''}\subset\tilde{M}$ which has no points in M. We take this segment and we construct an equidistant-curve e, circles C_1, C_2 and $Q(r)$ as in the case I. By M' denote the closed set $\tilde{M}$ without the open interval $\overline{x'x''}$. Then (12) is valid and therefore

$$\mu\tilde{M}_r - \mu M_r \geqslant \mu Q(r) \to +\infty \text{ as } r \to +\infty.$$

Q.E.D.

It is interesting to note that the Euclidean plane is an exceptional case in which $\mu\tilde{M}_r - \mu M_r \to 0$ as $r \to \infty$. In order to see this we consider in the n-dimensional Euclidean space a set $M = \{x,y\}$ consisting of two points x,y such that $d(x,y) = 2a > 0$. If μS denotes the n-dimensional measure (volume) of S and

$$S_r = \{z \mid d(z, S) \leqslant r,\ r > 0\}$$

the n-dimensional sphere around S with radius r, then $\mu\tilde{M}_r - \mu M_r$ behaves as $r^{n-3} (n \geqslant 2)$ for $r \to \infty$. Indeed if $V_{n-1}(\rho)$ denotes the volume of $(n-1)$-dimensional sphere with radius ρ, then $V_{n-1}(\rho) = \alpha_{n-1}\rho^{n-1}$ with a constant α_{n-1}. Now for $r > a$ we have:

$$\mu\tilde{M}_r - \mu M_r = 2a\ V_{n-1}(r) - 2\int_0^a V_{n-1}(\sqrt{r^2 - t^2})\,dt =$$

$$= 2\,\alpha_{n-1}\,[a\,r^{n-1} - \int_0^a (r^2 - t^2)^{\frac{n-1}{2}}\,dt] =$$

$$= 2\,\alpha_{n-1}\,r^n \left[\frac{a}{r} - \int_0^{a/r} (1 - t^2)^{\frac{n-1}{2}}\,dt\right] =$$

$$= \alpha_{n-1}(n-1)a^3 r^{n-3}\left[\frac{1}{3} - \frac{n-3}{20}\left(\frac{a}{r}\right)^2 + \dots\right],$$

i. e.

(15) $$\mu\tilde{M}_r - \mu M_r = O(r^{n-3}).$$

Probably (15) is valid for every bounded and closed set M in the n-dimensional $(n \geqslant 2)$ space which does not contain the boundary of $\tilde{M}$. This would be interesting to prove at least in the case $n=3$, because it seems that in this case $\lim_{r\to\infty} (\mu\tilde{M}_r - \mu M_r)$ exists, that this limit is finite but in general (even for a continuum M) different from zero.

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REFERENCE:

[1] G. Fast — The area of a generalized circle as a function of its radius I, II., *Fund. Math.* **46** (1958), (137—146) and (147—163).