THE SEQUENCE-TO-FUNCTION ANALOGUES TO QUASI-HAUSDORFF TRANSFORMATIONS[1]

AMNON JAKIMOVSKI (Jerusalem)

1. INTRODUCTION

In a recent paper [2] it was shown how to obtain transformations which transform sequences into functions and which might be thought of as the analogues to the Hausdorff transformations which transform sequences into sequences.

In this paper we shall obtain and investigate a class of new transformations (the K-transformations) which might be thought of as the sequence-to-function analogues to the quasi-Hausdorff transformations which transform sequences into sequences.

Suppose A and B are two transformations. Denote by AB their product which associates with a given sequence (or function) the A-transform of its B-transform. The following problem was and is being investigated. When and for which pairs A, B of transformations does A-summability imply AB-summability? We shall see that by means of our new K-transformations it will be possible to obtain some known results of the above type concerning products AB of transformations when B is a quasi-Hausdorff transformation as particular cases of some general theorems proved in this paper.

2. THE $[K, \mu_n]$ TRANSFORMATIONS

In this paper for all sequences the index denoting the order of terms assumes the values $0, 1, 2, \ldots$

[1] The research reported in this paper was sponsored in part by the Air Force Office of Scientific Research, Air Research and Development Command, U. S. A. Air Force, through its European Office.

For a given sequence $\{\mu_n\}$ the r-th difference $\Delta^r \mu_n$ is defined by

$$\Delta^r \mu_n = \sum_{p=0}^{n} (-1)^p \binom{n}{p} \mu_{n+p}, \quad r=0, 1, 2, \ldots$$

The quasi-Hausdorff transform (the $[QH, \mu_n]$) $\{t_n\}$ of a sequence $\{s_n\}$ by means of a fixed sequence $\{\mu_n\}$ is defined as the transform

$$t_n = \sum_{m=n}^{\infty} \binom{m}{n} (\Delta^{m-n} \mu_n) s_m, \quad n \geqslant 0 \tag{2.1}$$

(see [1], p. 277, § 11.19). Now it is easy to see that the transform $\{t_n\}$ defined by (2.1) is formally the same as the linear transform of $\{s_n\}$ of the form

$$t_n = \sum_{m=0}^{\infty} c_{nm} s_m, \quad n \geqslant 0, \tag{2.2}$$

satisfying

$$(n+1)\ (t_n - t_{n+1}) = \sum_{m=0}^{\infty} (m+1) c_{nm} (s_m - s_{m+1}), \quad n \geqslant 0; \tag{2.3}$$

that is, as the linear transformations of the form (2.2) commutative with the transformation defined by

$$t_n = (n+1)\ (s_n - s_{n+1}), \quad n \geqslant 0. \tag{2.4}$$

If we are trying to find a class of transformations which might be looked upon as the sequence-to-function analogues to the quasi-Hausdorff transformations we may proceed heuriestically in the following way.

We are looking for linear transformations

$$t(x) = \sum_{m=0}^{\infty} c_m(x) s_m \tag{2.5}$$

which should satisfy the following continuous analogues to (2.3)

$$-(x+1)\, t'(x) = \sum_{m=0}^{\infty} (m+1) c_m(x)\ (s_m - s_{m+1}). \tag{2.6}$$

That is, we want to obtain the general form of the functions $c_m(x)$.

All our calculations now will be formal ones. We have, by (2.5)

$$-(x+1)\, t'(x) = \sum_{m=0}^{\infty} -(x+1)\, c_m'(x) s_m, \tag{2.7}$$

$$\sum_{m=0}^{\infty} (m+1) c_m(x)\ (s_m - s_{m+1}) = c_0(x) s_0 + \sum_{m=1}^{\infty} [(m+1) c_m(x) - m c_{m-1}(x)] s_m. \tag{2.8}$$

Combining (2.6), (2.7) and (2.8) we obtain

$$(2.9)\qquad \sum_{m=0}^{\infty} -(x+1)\, c_m'(x)\, s_m = c_0(x)\, s_0 + \sum_{m=1}^{\infty} [\,(m+1)\, c_m(x) - m c_{m+1}(x)\,]\, s_m .$$

Since $\{s_n\}$ is an arbitrary sequence, we obtain, by equating the coefficients of s_m, the equations

$$(2.10)\qquad -(x+1)\, c_m'(x) = (m+1)\, c_m(x) - m c_{m-1}(x), \qquad m \geqslant 0.$$

In order to solve the equations (2.10) write, as we may,

$$(2.11)\qquad c_m(x) = (x+1)^{-(m+1)}\, d_m(x).$$

Substituting (2.11) in (2.10) we obtain the following equations

$$(2.12)\qquad \begin{cases} d_0'(x) = 0 \\ d_m'(x) = m\, d_{m-1}(x), \quad m > 0. \end{cases}$$

By mathematical induction we obtain the solution of (2.12)

$$d_m(x) = \sum_{p=0}^{m} \binom{m}{p} c_p\, x^{m-p}, \quad m \geqslant 0,$$

where $\{c_p\}$ is an arbitrary sequence of numbers. Therefore

$$(2.13)\qquad c_m(x) = (x+1)^{-(m+1)} \sum_{p=0}^{m} \binom{m}{p} c_p\, x^{m-p}, \qquad m \geqslant 0.$$

Thus the solution of (2.6) is

$$(2.14)\qquad t(x) = \sum_{m=0}^{\infty} s_m\, (x+1)^{-(m+1)} \sum_{p=0}^{m} \binom{m}{p} c_p\, x^{m-p}.$$

It is easy to see that

$$(2.15)\qquad \sum_{p=0}^{m} \binom{m}{p} c_p\, x^{m-p} = \sum_{p=0}^{m} (-1)^p \binom{m}{p} (\Delta^p c_0)\ (x+1)^{n-p}.$$

Therefore (2.14) might be written in the form

$$t(x) = (x+1)^{-1} \sum_{m=0}^{\infty} s_m \sum_{p=0}^{m} (-1)^p \binom{m}{p} (\Delta^p c_0)\ (x+1)^{m-p}.$$

Denote by $\{\mu_n\}$ the sequence $\{\Delta^n c_0\}$ and replace $x+1$ by x. Then we obtain the following class of transformations.

DEFINITION: *Given a fixed sequence* $\{\mu_n\}$, *we define the* $[K, \mu_n]$ *transform* $\{t_n\}$ *of a sequence* $\{s_n\}$ *by*

$$t(x) = x^{-1} \sum_{n=0}^{\infty} s_n \sum_{m=0}^{n} (-1)^m \binom{n}{m} \mu_m x^{-m}, \qquad x > x_0 > 0.$$

We say that $\{s_n\}$ *is summable* $[K, \mu_n]$ *to* s *if* $\lim\limits_{x \uparrow \infty} t(x) = s$.

Example 1 *:* If $\mu_n = a^{n+1} (a > 0)$ then the $[K, \mu_n]$ transform is the Abel transform expressed in the form

$$t(x) = (ax^{-1}) \sum_{n=0}^{\infty} s_n (1 - ax^{-1})^n, \qquad x \uparrow \infty,\ x > a.$$

Example 2*:* For any sequence $\{\mu_n\}$ and a constant $c > 0$ the $[K, \lambda_n = c^{n+1} \mu_n]$ transformation is essentially the same as the $[K, \mu_n]$ transformation. For, if the $[K, \mu_n]$, $[K, \lambda_n]$ transforms are, respectively, $t(x)$ and $t_c(x)$ then, as is easily seen, $t_c(x) = t(xc^{-1})$.

3. *REGULARITY OF THE* $[K, \mu_n]$ *TRANSFORMATIONS*

The main result of this section is

THEOREM (3.1): *If* $\mu_n = \int_0^1 t^{n+1} d\alpha(t)$ *(for* $n \geqslant 0$*), where* $\alpha(t)$ *is a function of bounded variation in the closed interval* $[0,1]$ *then the* $[K, \mu_n]$ *transformation is convergence preserving (that is it transforms convergent sequences into convergent transforms). The* $[K, \mu_n]$ *transformation is regular (that is it transforms convergent sequences into convergent transforms having the same limit) if, and only if,* $\alpha(1) - \alpha(0) = 1$.

Proof: We shall prove the second part of the theorem only. The proof of the first part of the theorem is exactly the same. First we prove the sufficiency part of the second part of the theorem. We have to show that the following three conditions are satisfied.

$$\overline{\lim_{x \uparrow \infty}}\ x^{-1} \sum_{n=0}^{\infty} \left| \sum_{m=0}^{n} (-1)^m \binom{n}{m} \mu_m x^{-m} \right| < \infty, \tag{3.1}$$

$$\lim_{x \uparrow \infty} x^{-1} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^m \binom{n}{m} \mu_m x^{-m} = 1, \tag{3.2}$$

$$\lim_{x \uparrow \infty} x^{-1} \sum_{m=0}^{n} (-1)^m \binom{n}{m} \mu_m x^{-m} = 0 \qquad \text{for } n = 0, 1, 2, \ldots \tag{3.3}$$

Now we have, for $x > 1$,

$$x^{-1}\sum_{n=0}^{\infty}\left|\sum_{m=0}^{n}(-1)^m\binom{n}{m}\mu_m x^{-m}\right| =$$

$$=\sum_{n=0}^{\infty}\left|\int_0^1 tx^{-1}\sum_{m=0}^{n}(-1)^m\binom{n}{m}t^m x^{-m}\,d\alpha(t)\right| =$$

$$=\sum_{n=0}^{\infty}\left|\int_0^1 tx^{-1}(1-tx^{-1})^n\,d\alpha(t)\right|$$

$$\leqslant\int_0^1 tx^{-1}\sum_{n=0}^{\infty}(1-tx^{-1})^n\,|d\alpha(t)| = \int_0^1 |d\alpha(t)|.$$

This shows that (3.1) is satisfied. We have, also,

$$x^{-1}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^m\binom{n}{m}\mu_m x^{-m} = \int_0^1 tx^{-1}\sum_{n=0}^{\infty}(1-tx^{-1})^n\,d\alpha(t)$$

$$=\int_0^1 d\alpha(t) = \alpha(1)-\alpha(0) = 1.$$

This shows that (3.2) is satisfied, too. Now

$$\left|x^{-1}\sum_{m=0}^{n}(-1)^m\binom{n}{m}\mu_m x^{-m}\right| = \left|\int_0^1 tx^{-1}(1-tx^{-1})^n\,d\alpha(t)\right|$$

$$\leqslant x^{-1}\int_0^1 t\,|d\alpha(t)| \to 0 \quad \text{as} \quad x\uparrow\infty.$$

Therefore (3.3) is also satisfied. The same argument proves the necessity part of the second part of the theorem. *Q. E. D.*

Example 2 of § 2 shows that the conclusions of Theorem (3.1) remain valid if

$$\mu_n = c^{n+1}\int_0^1 t^{n+1}\,d\alpha(t), \tag{3.4}$$

where c is any fixed positive constant and $\alpha(t)$ is of bounded variation in $0\leqslant t\leqslant 1$.

An open question is whether (3.1) is also a necessary condition for the convergence preserving property of the $[K, \mu_n]$ transformation.

4. THE PRODUCT OF THE $[K, \mu_n]$ AND QUASI-HAUSDORFF TRANSFORMATIONS

The main result of this section is

THEOREM (4.1): *Let $[K, c_p]$ be a convergence preserving transformation. If $\{s_n\}$ is summable $[K, c_p]$ to s and bounded then any regular $[QH, \mu_n]$ transform $\{t_n\}$ of $\{s_n\}$ is summable $[K, c_p]$ to s.*

Proof: It is known (see [4] and [3]) that a $[QH, \mu_n]$ transformation is regular if, and only if,

$$\mu_n = \int_0^1 t^{n+1}\, d\alpha(t), \quad n \geqq 0, \tag{4.1}$$

where $\alpha(t)$ is of bounded variation in $0 \leqslant t \leqslant 1$. We shall suppose therefore that $\{\mu_n\}$ has the representation (4.1). From the fact that $\{s_n\}$ is bounded (say $|s_n| \leqslant C$) we obtain for $0 \leqslant u \leqslant 1$

$$u^{n+1} \sum_{m=n}^{\infty} \binom{m}{n} (1-u)^{m-n} |s_m| \leqslant C \qquad \text{for} \quad n \geqslant 0.$$

From the convergence preserving property of the $[K, c_p]$ transformation we have

$$x^{-1} \sum_{n=0}^{\infty} \left| \sum_{p=0}^{n} (-1)^p \binom{n}{p} c_p\, x^{-p} \right| \leqslant K < \infty \qquad \text{for} \quad x \geqslant x_0 > 0.$$

Therefore we have for $0 < u \leqslant 1$

$$x^{-1} \sum_{n=0}^{\infty} \left\{ \sum_{p=0}^{n} (-1)^p \binom{n}{p} c_p\, x^{-p} \right\} u^{n+1} \sum_{m=n}^{\infty} \binom{m}{n} (1-u)^{m-n} s_m =$$

$$= x^{-1} \sum_{m=0}^{\infty} s_m \sum_{n=0}^{m} \binom{m}{n} (1-u)^{m-n} u^{n+1} \sum_{p=0}^{n} (-1)^p \binom{n}{p} c_p\, x^{-p}$$

$$= x^{-1} \sum_{m=0}^{\infty} s_m \sum_{p=0}^{m} (-1)^p c_p\, x^{-p} \sum_{n=p}^{m} \binom{m}{n} \binom{n}{p} (1-u)^{n-m} u^{n+1}$$

$$= x^{-1} \sum_{m=0}^{\infty} s_m \sum_{p=0}^{m} (-1)^p \binom{m}{p} c_p\, x^{-p} \sum_{n=p}^{m} \binom{m-p}{n-p} (1-u)^{n-n} u^{n+1}$$

$$= x^{-1} \sum_{m=0}^{\infty} s_m \sum_{p=0}^{m} (-1)^p \binom{m}{p} c_p\, x^{-p} \sum_{r=0}^{m-p} \binom{m-p}{r} (1-u)^{(m-p)-r}\, u^{r+p+1}$$

$$= (ux^{-1}) \sum_{m=0}^{\infty} s_m \sum_{p=0}^{m} (-1)^p \binom{m}{p} c_p\, (ux^{-1})^p .$$

Hence

$$\int_0^1 \left\{ (ux^{-1}) \sum_{m=0}^{\infty} s_m \sum_{p=0}^{m} (-1)^p \binom{m}{p} c_p\, (ux^{-1})^p \right\} d\alpha(u) =$$

$$= \int_0^1 \left\{ x^{-1} \sum_{n=0}^{\infty} \left[\sum_{p=0}^{n} (-1)^p \binom{n}{p} c_p\, x^{-p} \right] u^{n+1} \sum_{m=n}^{\infty} \binom{m}{n} (1-u)^{m-n} s_m \right\} d\alpha(u)$$

$$= x^{-1} \sum_{n=0}^{\infty} \left[\sum_{p=0}^{n} (-1)^p \binom{n}{p} c_p\, x^{-p} \right] \int_0^1 u^{n+1} \sum_{m=n}^{\infty} \binom{m}{n} (1-u)^{m-n} s_m\, d\alpha(u)$$

$$= x^{-1} \sum_{n=0}^{\infty} \left[\sum_{p=0}^{n} (-1)^p \binom{n}{p} c_p\, x^{-p} \right] \sum_{m=n}^{\infty} \binom{m}{n} (\Delta^{m-n} \mu_m)\, s_m$$

$$= x^{-1} \sum_{n=0}^{\infty} t_n \sum_{p=0}^{n} (-1)^p \binom{n}{p} c_p\, x^{-p} ,$$

Since it is quite easy to see that the integral on the left hand side is a regular transform of the $[K, c_p]$ transform of $\{s_n\}$ it follows from the fact that $\{s_n\}$ is summable $[K, c_p]$ to s that $\{t_n\}$ is also summable $[K, c_p]$ to s. *Q. E. D.*

The special case $[K, c_p] \equiv$ Abel's transformations in Theorem (4.1) was proved in [5].

From the proof of Theorem (4.1) it follows that the following theorem is also true.

THEOREM (4.2): *Let the $[K, c_p]$ transformation be covergence preserving. Suppose that for a given sequence $\{s_n\}$ we have*

$$x^{-1} \sum_{n=0}^{\infty} \left\{ u^{n+1} \sum_{m=n}^{\infty} \binom{m}{n} (1-u)^{m-n} |s_m| \right\} \left| \sum_{p=0}^{n} (-1)^p \binom{n}{p} c_p\, x^{-p} \right| \leqslant F(x) < \infty$$

for $0 < u \leqslant 1$ and $x \geqslant x_0 > 0$. If $\{s_n\}$ is summable $[K, c_p]$ to s then any regular quasi-Hausdorff transform $\{t_n\}$ of $\{s_n\}$ is summable $[K, c_p]$ to s, too.

It is easy to see that for $c_p = a^{p+1}\ (a > 0)$ and $\mu_n = b^{n+1}\ (0 < b < 1)$ the suppositions of the last theorem are satisfied. This special case is due to O. Szász [6].

By using the argument used in proving Theorem (4.1) and the argument used in proving Theorem (6.8) in [2] we obtain

THEOREM (4.3): *Let* $g(x)$ *be a real non-negative and monotonic increasing function for* $x > 0$ *and such that for each fixed* ε, $0 < \varepsilon < 1$,

$$\lim_{x \uparrow \infty} \frac{g(\varepsilon x)}{g(x)} = 1.$$

Let $[K, c_p]$ *be a convergence preserving transformation. Let* $\alpha > 0$ *be a fixed number. If for a bounded sequence* $\{s_n\}$ *we have*

$$\lim_{x \uparrow \infty} \frac{1}{x^{\alpha+1} g(x)} \sum_{n=0}^{\infty} s_n \sum_{m=0}^{\infty} (-1)^m \binom{n}{m} c_m x^{-m} = s$$

then for any regular $[QH, \beta(t)] \left(\mu_n = \int_0^1 t^{n+1} d\beta(t)\right)$ *transform* $\{t_n\}$ *of* $\{s_n\}$ *we have*

$$\lim_{x \uparrow \infty} \frac{1}{x^{\alpha+1} g(x)} \sum_{n=0}^{\infty} t_n \sum_{m=0}^{\infty} (-1)^m \binom{n}{m} c_m x^{-m} = s \cdot \int_0^1 t^{\alpha} d\beta(t).$$

REFERENCES

[1] G. H. Hardy — Divergent series. Oxford University Press, 1949.

[2] A. Jakimovski — The sequence-to-function analogues to Hausdorff transformations. *The Bulletin of the Research Council of Israel* **8F** (1960), 135—154.

[3] ———— Some remarks on the moment problem of Hausdorff. *Journal London Math. Soc.* **33** (1958), 1—14.

[4] M. S. Ramanujan — Series-to-series quasi-Hausdorff transformations. *Journal Indian Math. Soc.* **17** (1953), 347—53.

[5] ———— Theorems on the product of quasi-Hausdorff and Abel transforms. *Math. Zeit.* **64** (1956), 442—447.

[6] O. Szász — On the product of two summability methods. *Ann. Soc. Polonaise de Math.* **25** (1952/3), 75—84.