

BUCKLING PROBLEM OF RIGHT - ANGLED ISOSCELES TRIANGULAR PLATES

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SUMMARY — In the first part, the anti-symmetrical buckling of a simply supported triangular plate under the combining action of compression and shear is investigated. In the second part, the case when one side is clamped and acted only by the compression is treated by Taylor's method.

I. SIMPLE SUPPORTED PLATE UNDER THE ACTION OF UNIFORM COMPRESSION AND SHEARING FORCE

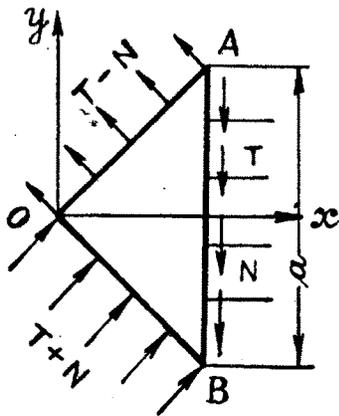


fig. 1

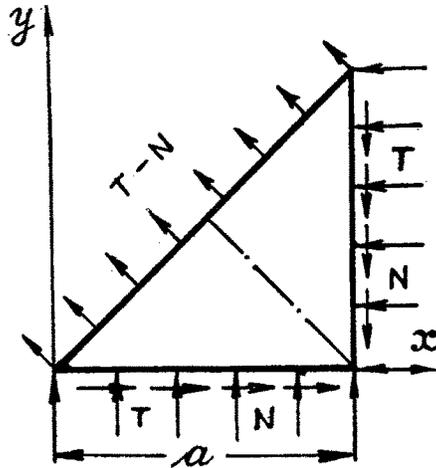


fig. 2

The total energy due to the strain of the triangular plate OAB (fig. 1.) under the action of uniform compression N and shearing force T is re-

presented by

$$T_v = \frac{1}{2} \iint \left\{ N \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] + 2T \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right\} dx dy + \quad (1)$$

$$+ \frac{D}{2} \iint \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1-\mu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy$$

where, w — deflection of plate at point (x, y) , D — its flexural rigidity, μ — Poisson's ratio.

The stability of the plate requires the minimalization of the strain energy, i. e. $\delta T_v = 0$. This condition imposed on the Eq. (1), gives:

$$\iint \left[D \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + N \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + 2T \frac{\partial^2 w}{\partial x \partial y} \right] \delta w dx dy = 0 \quad (2)$$

in which the integration is carried on the area of the plate, δw being an infinitesimal variation of the deflection satisfying the following boundary conditions of our problem

$$x = \frac{a}{2} \quad w, \quad \frac{\partial^2 w}{\partial x^2} = 0, \quad (3a)$$

$$x = \pm y \quad w, \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0. \quad (3b)$$

The buckling surface is sufficiently expressed by the following double trigonometric series satisfying the boundary condition (3a, 3b) term by term

$$w = \left[\sum_{m=2,6,\dots}^{\infty} \sum_{n=4,8,\dots}^{\infty} a_{mn} \left(\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} - \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{a} \right) + \sum_{p=1,5,\dots}^{\infty} \sum_{q=3,7,\dots}^{\infty} \left(\cos \frac{p\pi x}{a} \cos \frac{q\pi y}{a} - \cos \frac{q\pi x}{a} \cos \frac{p\pi y}{a} \right) \right] \quad (4)$$

in which the part of the sines describe the anti-symmetrical deflections, while the part of the cosines the symmetrical deflections. Taking subsequently

$$\delta w = \varepsilon_{ij} \left(\sin \frac{i\pi x}{a} \sin \frac{j\pi y}{a} - \sin \frac{j\pi x}{a} \sin \frac{i\pi y}{a} \right), \quad (5a)$$

$$(i = 2, 6, 10, \dots; j = 4, 8, 12, \dots)$$

$$\delta w = \varepsilon_{st} \left(\cos \frac{s\pi x}{a} \cos \frac{t\pi y}{a} - \cos \frac{t\pi x}{a} \cos \frac{s\pi y}{a} \right), \quad (5b)$$

$$(s = 1, 5, 9, \dots, t = 3, 7, 11, \dots)$$

where ε_{ij} , ε_{st} are small quantities and utilizing the following relations:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} \sin \frac{p\pi x}{a} \sin \frac{i\pi x}{a} dx = \begin{cases} 0, & \text{when } m \neq i, \\ \frac{a}{2}, & \text{when } m = i, \end{cases} \quad (6a)$$

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} \sin \frac{p\pi x}{a} \sin \frac{i\pi x}{a} dx = \begin{cases} \frac{2a}{\pi} \frac{i}{p^2 - i^2}, & \text{when } \frac{pi-1}{2} \text{ is even,} \\ -\frac{2a}{\pi} \frac{i}{p^2 - i^2}, & \text{when } \frac{p+i-1}{2} \text{ is odd.} \end{cases} \quad (6c)$$

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos \frac{p\pi x}{a} \cos \frac{s\pi x}{a} dx = \begin{cases} 0, & \text{when } p \neq s, \\ \frac{a}{2}, & \text{when } p = s, \end{cases} \quad (7a)$$

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} \cos \frac{m\pi x}{a} \cos \frac{s\pi x}{a} dx = \begin{cases} \frac{2a}{\pi} \frac{s}{s^2 - m^2}, & \text{when } \frac{m+s-1}{2} \text{ is even} \\ -\frac{2a}{\pi} \frac{s}{s^2 - m^2}, & \text{when } \frac{m+s-1}{2} \text{ is odd,} \end{cases} \quad (7c)$$

the following infinite, homogeneous, simultaneous equations are obtained

$$a_{mn}(m^2 + n^2)(m^2 + n^2 - k_N) \lambda - \sum_p \sum_q b_{pq} \frac{mnpq(m^2 - n^2)(p^2 - q^2)}{(m^2 - p^2)(m^2 - q^2)(n^2 - p^2)(n^2 - q^2)} = 0 \quad (8a)$$

and

$$b_{pq}(\rho^2 + q^2)(\rho^2 + q^2 - k_N) \lambda - \sum_m \sum_n a_{mn} \frac{mnpq(m^2 - n^2)(p^2 - q^2)}{(m^2 - p^2)(m^2 - q^2)(n^2 - p^2)(n^2 - q^2)} = 0 \quad (8b)$$

in which

$$\lambda = \frac{\pi^4}{32} \frac{D}{a^2 T}, \quad k_N = \frac{a^2 N}{\pi^2 D}, \quad k_T = \frac{a^2 T}{\pi^2 D}.$$

Loss of stability, a_{mn} , $b_{pq} \neq 0$ gives from Eq (8)

b_{12}	a_{24}	b_{53}	a_{64}	b_{57}	...	
$10(10 - k_N)\lambda$	$\frac{256}{175}$	0	$\frac{256}{2205}$	0	...	
$\frac{256}{175}$	$20(20 - k_N)\lambda$	$-\frac{512}{147}$	0	$-\frac{256}{891}$...	
0	$-\frac{512}{147}$	$34(34 - k_N)\lambda$	$\frac{12800}{2079}$	0	...	$= 0 \quad (9)$
$\frac{256}{2205}$	0	$\frac{12800}{2079}$	$52(52 - k_N)\lambda$	$-\frac{44800}{4719}$...	
0	$-\frac{256}{891}$	0	$-\frac{44800}{4719}$	$74(74 - k_N)\lambda$...	
...	

Let us investigate a few special cases.

1) When the plate supports only the uniform compression ($k_T = 0$), the fundamental form of the symmetrical buckling is (supposing $k_N = 10$)

$$w = b_{12} \left(\cos \frac{\pi x}{a} \cos \frac{3\pi y}{a} - \cos \frac{3\pi x}{a} \cos \frac{\pi y}{a} \right) \quad (10 a)$$

that of anti-symmetrical buckling is (supposing $k_N = 20$)

$$w = a_{24} \left(\sin \frac{2\pi x}{a} \sin \frac{4\pi y}{a} - \sin \frac{4\pi x}{a} \sin \frac{2\pi y}{a} \right). \quad (10 b)$$

2) When the plate supports only the uniform shearing force ($k_N = 0$), we annul subsequently the determinants of 2 rows and 2 columns ($\Delta_2 = 0$) of 3 rows and 3 columns ($\Delta_3 = 0$) etc., and find the maximum roots of the absolute value of λ corresponding to the minimum critical force,

	$\Delta_2 = 0$	$\Delta_3 = 0$	$\Delta_4 = 0$	$\Delta_5 = 0$	approximate value
$k_T =$	$\pm 42,16$	$\pm 34,53$	$\pm 33,76$	$\pm 33,74$	$\pm 33,6$

the double sign expresses the fact that the critical force is independent of the shearing force.

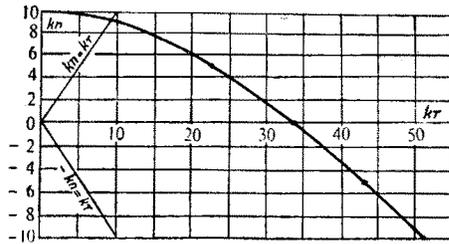
3) Another special case is the equality in magnitude of the shearing force and the compression ($k_N = k_T = k$) which corresponds to the case that the right-angled side is under simple compression. We have found $k = 9,08$; this is quite near to the result $k = 9,11$ obtained by W. Burchard [1] by means of the relaxation method.

Under the general case when $-10 \leq k_N \leq 10$ the results we have obtained are shown in Table 1 and figure 3.

 Table 1. Relation of k_N and k_T

$k_N =$	+ 10	+ 9,08	+ 5	0	- 5	- 10
$k_T =$	0	$\pm 9,08$	$\pm 22,5$	$\pm 33,6$	$\pm 43,1$	$\pm 51,6$
Order of determinant	—	4 th	5 th	5 th	5 th	5 th

When $-k_N = k_T$ which corresponds to the case that the right-angled side is subjected to simple tension, there is evidently no buckling, therefore the curve $k_N - k_T$ cannot intersect with the straight line $-k_N = k_T$.


 fig. 3. Relation of k_N and k_T

When the uniform shearing force is acting on the right angled sides (fig. 2). Klitchieff [2] and Wittrick [3] investigated this problem by means of the following double trigonometric series

$$w = \sum_{m=2,4,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} a_{mn} \left(\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} - \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{a} \right). \quad (11)$$

It should be pointed out that the Eq. (11) can describe only the buckling symmetrical to the median of the hypotenuse, while the Eq. (4) should be used to describe the anti-symmetrical buckling.

II. ONE SIDE CLAMPED AND UNDER THE ACTION OF UNIFORM COMPRESSION

We start from the equation of the small deflection of thin plate:

$$D \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + N \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = 0. \quad (12)$$

Introducing the following dimensionless parameters

$$x = \frac{2a\xi}{\pi}, \quad y = \frac{2a\eta}{\pi}, \quad k = \frac{4a^2 N}{\pi^2 D} \quad (13)$$

the Eq. (12) is transformed as

$$\left(\frac{\partial^4 w}{\partial \xi^4} + 2\frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 w}{\partial \eta^4}\right) + k\left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2}\right) = 0. \quad (14)$$

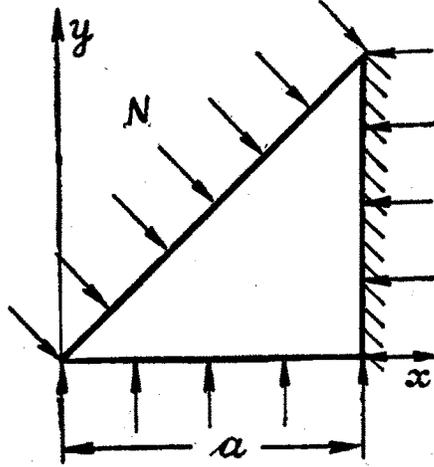


fig. 4.

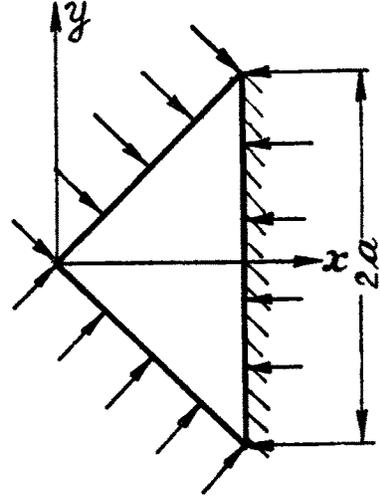


fig. 5.

When one right-angled side is clamped (fig. 4), the boundary conditions are

$$\xi = \frac{\pi}{2}, \quad w, \quad \frac{\partial w}{\partial \xi} = 0, \quad (15 a, 15 b)$$

$$\eta = \xi, \quad w, \quad \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = 0, \quad (15 c)$$

$$\eta = 0, \quad w, \quad \frac{\partial^2 w}{\partial \eta^2} = 0. \quad (15 d)$$

As the function $e^{\alpha \eta} \sin m \xi$ can satisfy the boundary condition (15a), where m is an even integer and α a constant determined by the following equation. Substituting this function into (14), we have

$$(\alpha^2 - m^2)(\alpha^2 - m^2 + k) = 0 \quad (16)$$

The Eq. (16) has

$$\left. \begin{array}{l} \text{two real roots} \quad \alpha = \pm m \\ \text{two imaginary roots} \quad \alpha = \pm i\sqrt{k - m^2} \text{ (when } m < \sqrt{k}) \\ \text{or two real roots} \quad \alpha = \pm \sqrt{m^2 - k} \text{ (when } m > \sqrt{k}) \end{array} \right\} \quad (17)$$

Now the buckling surface can be expressed as satisfying term by term the series of the function of the Eq. (14)

$$\begin{aligned}
 w = & \sum_{m < \sqrt{k}} A_m \left[\begin{aligned} & \left(\operatorname{sh} \frac{m\pi}{2} \sin \sqrt{k-m^2} \eta - \sin \frac{\pi}{2} \sqrt{k-m^2} \operatorname{sh} m\eta \right) \sin m\xi - \\ & - \left(\operatorname{sh} \frac{m\pi}{2} \sin \sqrt{k-m^2} \xi - \sin \frac{\pi}{2} \sqrt{k-m^2} \operatorname{sh} m\xi \right) \sin m\eta \end{aligned} \right] + \\
 & + \sum_{m > \sqrt{k}} A_m \left[\begin{aligned} & \left(\operatorname{sh} \frac{m\pi}{2} \operatorname{sh} \sqrt{m^2-k} \eta - \operatorname{sh} \frac{\pi}{2} \sqrt{m^2-k} \operatorname{sh} m\eta \right) \sin m\xi - \\ & - \left(\operatorname{sh} \frac{m\pi}{2} \operatorname{sh} \sqrt{m^2-k} \xi - \operatorname{sh} \frac{\pi}{2} \sqrt{m^2-k} \operatorname{sh} m\xi \right) \sin m\eta \end{aligned} \right]. \quad (18)
 \end{aligned}$$

Now only the condition (15 b) is left unsatisfied, arranging A_m such as

$$\left[\frac{\partial w}{\partial \xi} \right]_{\xi=\pi/2} = 0,$$

and using

$$\sin \alpha \eta = -\frac{4\alpha}{\pi} \cos \frac{\alpha\pi}{2} \left(\frac{\sin \eta}{\alpha^2-1^2} - \frac{\sin 3\eta}{\alpha^2-3^2} + \frac{\sin 5\eta}{\alpha^2-5^2} - \dots \right), \quad (19 a)$$

$$\operatorname{sh} \alpha \eta = \frac{4\alpha}{\pi} \operatorname{ch} \frac{\alpha\pi}{2} \left(\frac{\sin \eta}{\alpha^2+1^2} - \frac{\sin 3\eta}{\alpha^2+3^2} + \frac{\sin 5\eta}{\alpha^2+5^2} - \dots \right), \quad (19 b)$$

we obtain a set of infinite, homogeneous simultaneous equations

$$\frac{4}{\pi} \sum_{t=1,3,5,\dots}^{\infty} (-1)^{m/2} m A_m a_{mt} = 0 \quad (m=2, 4, 6, \dots) \quad (20)$$

where

$$\begin{aligned}
 a_{mt} = & \operatorname{sh} \frac{m\pi}{2} \sin \frac{\pi \sqrt{k-m^2}}{2} \left[\begin{aligned} & \sqrt{k-m^2} \operatorname{ctg} \frac{\pi \sqrt{k-m^2}}{2} \left(-\frac{1}{k-m^2-t^2} + \frac{1}{m^2-t^2} \right) - \\ & - m \operatorname{cth} \frac{m\pi}{2} \left(\frac{1}{m^2+t^2} + \frac{1}{m^2-t^2} \right) \end{aligned} \right] \\
 & \text{(when } m < \sqrt{k} \text{)} \quad (21 a)
 \end{aligned}$$

$$a_{mt} = \text{sh} \frac{m\pi}{2} \text{sh} \frac{\pi\sqrt{m^2-k}}{2} \left[\begin{aligned} & \sqrt{m^2-k} \text{cth} \frac{\pi\sqrt{m^2-k}}{2} \left(\frac{1}{m^2-k+t^2} + \frac{1}{m^2-t^2} \right) - \\ & - m \text{cth} \frac{m\pi}{2} \left(\frac{1}{m^2+t^2} + \frac{1}{m^2-t^2} \right) \end{aligned} \right] \quad (\text{when } m > \sqrt{k}) \quad (21 b)$$

Annuling their determinant coefficients, the approximate value $k=27,26$ ($\Delta_5=0$) is obtained

$$N_{cr} = 6,82 \frac{\pi^2 D}{a^2}.$$

If the hypotenuse is clamped (fig. 5), the boundary conditions are

$$\xi = \frac{\pi}{2} \quad w, \quad \frac{\partial w}{\partial \xi} = 0, \quad (22 a, 22 b)$$

$$\eta = \pm \xi \quad w, \quad \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} = 0, \quad (22 c)$$

the buckling surface is expressed as

$$w = \sum_{n < \sqrt{k}} B_n \left[\begin{aligned} & \left(\text{ch} \frac{n\pi}{2} \cos \sqrt{k-n^2} \eta - \cos \frac{\pi}{2} \sqrt{k-n^2} \text{ch} n\eta \right) \cos n\xi - \\ & - \left(\text{ch} \frac{n\pi}{2} \cos \sqrt{k-n^2} \xi - \cos \frac{\pi}{2} \sqrt{k-n^2} \text{ch} n\xi \right) \cos n\eta \end{aligned} \right] + \\ + \sum_{n > \sqrt{k}} B_n \left[\begin{aligned} & \left(\text{ch} \frac{n\pi}{2} \text{ch} \sqrt{n^2-k} \eta - \text{ch} \frac{\pi}{2} \sqrt{n^2-k} \text{ch} n\eta \right) \cos n\xi - \\ & - \left(\text{ch} \frac{n\pi}{2} \text{ch} \sqrt{n^2-k} \xi - \text{ch} \frac{\pi}{2} \sqrt{n^2-k} \text{ch} n\xi \right) \cos n\eta \end{aligned} \right]$$

where n is an odd integer.

Now only the condition (22b) is left unsatisfied; arranging B_n such as

$$\left[\frac{\partial w}{\partial \xi} \right]_{\xi=\pi/2} = 0$$

and using

$$\cos \beta \eta = \frac{4\beta}{\pi} \sin \frac{\beta \pi}{2} \left(\frac{1}{2\beta^2} - \frac{\cos 2\eta}{\beta^2 - 2^2} + \frac{\cos 4\eta}{\beta^2 - 4^2} - \dots \right) \quad (24a)$$

$$\operatorname{ch} \beta \eta = \frac{4\beta}{\pi} \operatorname{sh} \frac{\beta \pi}{2} \left(\frac{1}{2\beta^2} - \frac{\cos 2\eta}{\beta^2 + 2^2} + \frac{\cos 4\eta}{\beta^2 + 4^2} - \dots \right) \quad (24b)$$

we obtain

$$\frac{4}{\pi} \sum_{s=0,2,4,\dots}^{\infty} (-1)^{\frac{s}{2}} n B_n b_{ns} = 0 \quad (n=1,3,5,\dots) \quad (25)$$

where

$$b_{ns} = \operatorname{ch} \frac{n\pi}{2} \cos \frac{\pi \sqrt{k-n^2}}{2} \left[\begin{aligned} & \sqrt{k-n^2} \operatorname{tg} \frac{\pi \sqrt{k-n^2}}{2} \left(\frac{1}{k-n^2-s^2} - \frac{1}{n^2-s^2} \right) - \\ & - n \operatorname{th} \frac{n\pi}{2} \left(\frac{1}{n^2+s^2} + \frac{1}{n^2-s^2} \right) \end{aligned} \right] \quad (\text{when } n < \sqrt{k}) \quad (26a)$$

$$b_{ns} = \operatorname{ch} \frac{n\pi}{2} \operatorname{ch} \frac{\pi \sqrt{n^2-k}}{2} \left[\begin{aligned} & \sqrt{n^2-k} \operatorname{th} \frac{\pi \sqrt{n^2-k}}{2} \left(\frac{1}{n^2-k+s^2} + \frac{1}{n^2-s^2} \right) - \\ & - n \operatorname{th} \frac{n\pi}{2} \left(\frac{1}{n^2+s^2} + \frac{1}{n^2-s^2} \right) \end{aligned} \right] \quad (\text{when } n > \sqrt{k}) \quad (26a)$$

approximate value of k is got as 15,62 ($\Delta_5=0$),

$$N_{cr} = 7,81 \frac{\pi^2 D}{(a\sqrt{2})^2}$$

while Wittrick's result is 7,82 by the energy method.

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