ON SOME *REGULAR RINGS

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1. A regular ring A is a ring in which for every $a \in A$ there exists an element $x \in A$ such that

(1) $$axa = a.$$

We shall assume that A has a unit element 1.

A *regular ring is a regular ring A with an involution, i. e. with a map $x \to x^*$ of A onto A such that

(2) $$x^{**} = x, \quad (x+y)^* = x^* + y^*, \quad (xy)^* = y^* x^*$$

and such that

(A) $$x^* x = 0 \quad \text{implies} \quad x = 0.$$

An element $a \in A$ is called self-adjoint if $a^* = a$.

In this paper we shall study in some detail the *regular rings satisfying the stronger axiom:

(B) If $x^* x + y^* y + \ldots + u^* u = 0$, then $x = y = \ldots = u = 0$, where the number of elements $x, y, \ldots, u \in A$ is arbitrary.

It can be easily shown that a *regular ring satisfying the axiom (B) is an algebra over the field of rational numbers. Moreover, it is possible to define in a natural way the notion of boundedness of an element $a \in A$. The bounded elements of A have a norm and this norm possesses nearly the same properties as the norm in a C*-algebra.

An immediate consequence of the axiom (B) is:

LEMMA 1. *If $na = 0$, where n is an integer and $a \in A$, then $a = 0$.*

Here na signifies the sum $a + a + \ldots + a$ (n-times). From $a + a + \ldots + a = 0$ it follows that $a^* a + a^* a + \ldots + a^* a = 0$. Hence, by axiom (B), $a = 0$.

Let e_n be a solution of (1) for $a = n.\,1$ (n is a positive integer):

$$(n\cdot 1)\,e_n\,(n\cdot 1) = n\cdot 1 \quad \text{or} \quad n^2 e_n = n\cdot 1$$

Since $n\,(ne_n - 1) = 0$, it follows from lemma 1 that $ne_n = 1$. The element e_n is uniquely determined. For let us suppose that we have also $nf_n = 1$, Then $n\,(f_n - e_n) = 0$, thus $f_n = e_n$.

Let us therefore write $e_n = \frac{1}{n}\cdot 1$ and $me_n = \frac{m}{n}\cdot 1$ or simply $me_n = \frac{m}{n}$. where m is any positive or negative integer. We can easily verify that $n\cdot\left(\frac{m}{n}\cdot 1\right) = m\cdot 1$ and that the element $\frac{m}{n}\cdot 1$ depends only on the value of the fraction m/n. The ring A contains therefore a field isomorphic to the field of rational numbers.

Write $\left(\frac{m}{n}\cdot 1\right)a = \frac{m}{n}\,a$. It is easy to see that the following rules

a) $\alpha\,(a+b) = \alpha a + \alpha b$, b) $(\alpha+\beta)\,a = \alpha a + \beta a$,

c) $\alpha\,(\beta a) = (\alpha\beta)\,a$, d) $(\alpha a)\,(\beta b) = (\alpha\beta)\,ab$,

are valid for any two rational numbers α, β and any two elements $a, b \in A$ We have thus established:

LEMMA 2. *A *regular ring in which the axiom* (B) *holds is an algebra over the field of rational numbers.*

2. An element $p \in A$ will be called *positive* if it can be written in the form $p = x^* x + y^* y + \ldots + u^* u$, where $x, y, \ldots, u \in A$. A positive element p is self-adjoint. Denote by P the set of all positive elements of A. Evidently $1 = 1^* \cdot 1 \in P$, and, for any integer $m > 0$, we have also $\frac{m}{n} \in P$, since

$$\frac{m}{n} = \frac{mn}{n^2} = \frac{1}{n^2} + \frac{1}{n^2} + \ldots + \frac{1}{n^2}.$$

Further, $p, q \in P$ implies $p + q \in P$ and $\alpha p \in P$ for every rational number $\alpha > 0$. If $p \in P$ and $a \in A$, then $a^* pa \in P$. For let us write $p = x^* x + \ldots + u^* u$. Then $a^* pa = (xa)^* (xa) + \ldots + (ua)^* (ua)$. Thus $a^* pa \in P$. From axiom (B) it also follows that $u \in P$ and $-u \in P$ implies $u = 0$.

If a and b are self-adjoint and $a - b \in P$, we shall write $a > b$ or, equivalently, $b < a$.

LEMMA 3. *Let $p \in P$. Then there exists an element $u \in P$ such that $pup = p$.*

Proof. The regularity of A implies the existence of an $x \in A$ satisfying the equation $pxp = p$. Since $p^* = p$, we have also $px^*p = p$. Put $y = \frac{1}{2}(x + x^*)$. Then $y^* = y$ and $pyp = p$. Now, the element $u = ypy$ has the required properties: $u \in P$ and $pup = p$.

In particular: the inverse a^{-1} of a positive regular element a is positive.

We shall say that an element a is bounded from above if there exists a rational number ξ such that $a \leqq \xi$. In this case, $a = \xi - p$, where $p \in P$. Thus a is self-adjoint. Similarly is defined the boundedness from below. Let a be bounded from above. Denote by α the infimum of all numbers ξ such that $a \leqq \xi$. In this case also we shall write $a \leqq \alpha$, though the number α may be irrational.

DEFINITION. *An element $a \in A$ will be called bounded if there exists a rational number ξ such that $a^* a \leqq \xi$.*

If a is bounded, then $\xi = a^* a + p$, where $p \in P$. Let α be the greatest lower bound of all rational numbers ξ satisfying the inequality $a^* a \leqq \xi$. The positive square root of α will be called the *norm* of a and designated by $\sqrt{\alpha} = \|a\|$.

Take two elements $a, b \in A$ and any two rational numbers λ, μ. From

$$(\lambda a + \mu b)^* (\lambda a + \mu b) = p \in P.$$

we deduce

$$\lambda\mu (a^* b + b^* a) = p - \lambda^2 a^* a - \mu^2 b^* b. \tag{3}$$

Now, suppose a and b bounded. Let α, β be any two rational numbers so that $\alpha > \|a\|$ and $\beta > \|b\|$. Then

$$a^* a = \alpha^2 - p_1, \quad b^* b = \beta^2 - p_2, \quad p_1, p_2 \in P. \tag{4}$$

If we put $\lambda = \beta$, $\mu = \alpha$ in (3) we have

$$a^* b + b^* a = -2\alpha\beta + \frac{1}{\alpha\beta}(p + \beta^2 p_1 + \alpha^2 p_2) \geqq -2\alpha\beta. \tag{5}$$

Since α, β are any two rational numbers such that $\alpha > \|a\|$ and $\beta > \|b\|$, it follows from (5) that $-2\|a\| \, \|b\|$ is a lower bound of $a^* b + b^* a$. Similarly, taking $\lambda = \beta$, $\mu = -\alpha$, we deduce that $2\|a\|\,\|b\|$ is an upper bound of $a^* b + b^* a$. Thus

$$(6) \qquad -2\|a\|\,\|b\| \leqq a^* b + b^* a \leqq 2\|a\|\,\|b\|.$$

Now, consider the sum $a+b$ where a and b are bounded. From (6) we have

$$(a+b)^*(a+b) = a^* a + b^* b + a^* b + b^* a \leqq \|a\|^2 + \|b\|^2 + 2\|a\|\,\|b\| = (\|a\| + \|b\|)^2.$$

The number $(\|a\| + \|b\|)^2$ being an upper bound of $(a+b)^*(a+b)$, it follows that

$$(\mathrm{I}) \qquad \|a+b\| \leqq \|a\| + \|b\|.$$

Multiplying the first equality (4) on the left by b^* and on the right by b, we get

$$b^* a^* ab = \alpha^2 b^* b - b^* p_1 b = \alpha^2 \beta^2 - \alpha^2 p_2 - b^* p_1 b.$$

It follows that $(ab)^*(ab) \leqq \alpha^2 \beta^2$; therefore

$$(\mathrm{II}) \qquad \|ab\| \leqq \|a\|\,\|b\|.$$

Finally, it is easy to see that

$$(\mathrm{III}) \qquad \|\alpha a\| = |\alpha|\,\|a\|$$

for every rational α and every bounded $a \in A$. Summing up the results just obtained we have:

LEMMA 4. *The sum and product of two bounded elements of A are bounded. The norm of bounded elements possesses the properties* (I), (II) *and* (III).

Therefore, the bounded elements form a subring A_1 of the ring A. A_1 is also an algebra over the field of rational numbers.

3. A self-adjoint idempotent of A is called a projection. Two projections e and f are equivalent if there exist elements $x \in eAf$ and $y \in fAe$ such that $xy = e$ and $yx = f$. The projections of a *regular ring form an orthocomplemented modular lattice.

LEMMA 5. *Projections of a *regular ring A satisfying the axiom* (B) *are bounded. If $e \neq 0$, then $\| e \| = 1$.*

Proof. Let e be a projection. Since $e^* e = e = 1 - (1 - e)^2$, we have $\| e \| \leqq 1$. Suppose now that $e = \alpha - p$, where $p \in P$ and $0 \leqq \alpha \leqq 1$. Multiplication with e on both sides gives $e = \alpha e - epe$, thus $(1 - \alpha) e + epe = 0$. Therefore, by axiom (B), $(1 - \alpha) e = 0$. If $e \neq 0$, then $\alpha = 1$. It follows that $\| e \| = 1$.

LEMMA 6. *For any $a \in A$, the inverse of $1 + a^* a$ is bounded with a norm $\leqq 1$.*

The element $1 + a^* a$ is neither left nor right divisor of zero. This follows at once from axiom (B), since $(1 + a^* a) x = 0$ implies $x^* x + (ax)^* (ax) = 0$, and so $x = 0$. Therefore, the inverse $(1 + a^* a)^{-1}$ exists and it is positive. Now, the identity

$$[(1 + a^* a)^{-1}]^2 = 1 - 2(1 + a^* a)^{-1} (a^* a) (1 + a^* a)^{-1} - (1 + a^* a)^{-1} (a^* a)^2 (1 + a^* a)^{-1}$$

shows that $\| (1 + a^* a)^{-1} \| \leqq 1$.

LEMMA 7. *Let a be bounded. The element $a^* a - \lambda$ is regular for every rational number $\lambda > \| a \|^2$.*

It suffices to show that the difference $a^* a - \lambda$ is not divisor of zero when $\lambda > \| a \|^2$. Take a rational number α such that $\| a \|^2 < \alpha < \lambda$. Then $a^* a = \alpha - p$ with $p \in P$, since $\alpha > \| a \|^2$. Suppose that there exists an x satisfying $(a^* a - \lambda) x = 0$. From this we deduce

$$x^* (a^* a - \lambda) x = -(\lambda - \alpha) x^* x - x^* p x = 0.$$

Since $\lambda > \alpha$, it follows that $x = 0$.

LEMMA 8. *The adjoint a^* of a bounded element a is bounded, and $\| a^* \| = \| a \|$.*

Proof [2]. Let α be any rational number $> \| a \|$. Then $a^* a = \alpha^2 - p$, $p \in P$. From this we deduce

$$aa^* = \alpha^2 - \frac{1}{\alpha^2} (aa^* - \alpha^2)^2 - \frac{1}{\alpha^2} apa^*.$$

Therefore, $\|a^*\| \leqq \alpha$, the last two members on the right being positive. Since $\alpha > \|a\|$ is arbitrary, it follows that $\|a^*\| \leqq \|a\|$. If we change a into a^*, we get also $\|a\| \leqq \|a^*\|$. Hence, $\|a^*\| = \|a\|$.

Let us calculate the norm of a^*a, where a is bounded. Choose any rational number $\alpha > \|a^*a\|$. Then $(a^*a)^2 = \alpha^2 - p$, $p \in P$. From this we obtain

$$a^*a = \alpha - \frac{1}{2\alpha}(a^*a - \alpha)^2 - \frac{1}{2\alpha}p\,. \tag{7}$$

It follows that $\|a\|^2 \leqq \|a^*a\|$. On the other hand, by (II), we have $\|a^*a\| \leqq \|a^*\|\,\|a\| = \|a\|^2$. Therefore

$$\|a^*a\| = \|a\|^2\,. \tag{IV}$$

The ring A_1 of all bounded elements of A is self-adjoint and includes all projections of A. According to (I), (II), (III) and (IV) this ring resembles very closely a C*-algebra. Of course, A_1 is in general not regular. *If the ring A_1 is regular, then $A_1 = A$, therefore all elements of A are bounded.* Let namely a be an arbitrary element of A. By lemma 6, $(1+a^*a)^{-1}$ is bounded, so $(1+a^*a)^{-1} \in A_1$. Since A_1 is supposed to be regular, the inverse of any regular element of A_1 belongs to A_1. So $1+a^*a = [(1+a^*a)^{-1}]^{-1} \in A_1$. Therefore, $a^*a \in A_1$ and equality (7) shows that $a \in A_1$, too. Thus $A_1 = A$.

The norm in the ring A_1 has almost all the properties of the norm in a Banach algebra. Only the equation $\|x\| = 0$ in general does not imply $x = 0$. We shall call an element $x \in A_1$, such that $\|x\| = 0$, infinitely small. From (I), (II) and (IV) it follows that the infinitely small elements form a two sided *ideal of the ring A_1.

THEOREM 1. *If the ring A_1 is regular, that is if $A_1 = A$, then there do not exist infinitely small elements different from zero.*

P r o o f. Since A_1 is by assumption *regular, every principal right ideal in A_1 is generated by a projection. Therefore, there exists for every $a \in A_1$ a projection e, such that $aA_1 = eA_1$, where $e \neq 0$, if $a \neq 0$. Thus we can write $e = ax$ with $x \in A_1$. If $a \neq 0$, it follows from this that $1 = \|e\| = \|ax\| \leqq \|a\|\,\|x\|$. Therefore $\|a\| \neq 0$.

If $A_1 = A$, the norm has all the properties of a norm in a Banach algebra. But A is in general not topologically complete for this norm. If A is complete, then it is a C*-algebra and is also a finite dimensional algebra, according to a result of I. Kaplansky.

4. A *regular ring is said to be complete if its lattice of projections is complete. I. Kaplansky has recently demonstrated [1], that the lattice of projections of a complete *regular ring forms a continuous geometry in the sense of von Neumann.

THEOREM 2. *If A_1 is a complete *regular ring, then its lattice of projections is of type* I.

P r o o f. Suppose that, on the contrary, the lattice of projections of the ring A_1 were not of type I. Then there exists a projection e which can be split into the sum of two equivalent orthogonal projections $e=e_1+f_1$, where e_1 and f_1 have the same property, i.e. they can also be split into the sum of two orthogonal equivalent projections, and so on. Therefore, there exists a sequence of mutually orthogonal projections $e_1, e_2, e_3, \ldots$ such that $e_1>e_2>e_3>e_4>\ldots$. Hence, we can write $e_n=f_n+h_n$, where $f_n \perp h_n$, and $f_n \sim e_{n+1}$. The last equivalence implies the existence of elements $u_n \in f_n A_1 e_{n+1}$ and $v_n \in e_{n+1} A_1 f_n$ such that $u_n v_n = f_n$, $v_n u_n = e_{n+1}$. Since $f_n e_{n+1}=0$, we have $u_n^2 \in (f_n A_1 e_{n+1})\ (f_n A_1 e_{n+1}) = 0$. The elements $\varkappa u_n$ and $\frac{1}{\varkappa} v_n$ have the same property as u_n, v_n for every rational $\varkappa$. We may therefore suppose that $\lim_{n\to\infty} \| u_n \| = 0$.

Let us denote by B the set of all elements belonging to A_1 which commute with all projections $g_k=e_{2k-1}+e_{2k}$, $k=1,2,3,\ldots$ The set B is a complete *regular ring [1] and contains all g_k's. In the ring B the g_k's are central projections. Let g be the least upper bound of g_k's. The ring Bg, too, is complete and *regular. Moreover

$$u_{2k-1}\, g_k = g_k\, u_{2k-1} = u_{2k-1} \quad \text{and} \quad u_{2k-1}\, g_i = g_i\, u_{2k-1} = 0, \quad \text{if } i \neq 2k-1.$$

So $u_{2k-1} \in Bg_k$. According to lemma 12, [1], there exists in Bg an element u such that $ug_k=u_{2k-1}$ for all k's. But the ring A_1 is regular. Hence there exists an $x \in A_1$ satisfying the equality $uxu=u$. Multiplying both sides of this equality with g_k we get $u_{2k-1}\, x u_{2k-1} = u_{2k-1}$. It follows that $\| u_{2k-1} \| = \| u_{2k-1}\, x u_{2k-1} \| \leqq \| u_{2k-1} \|^2 \| x \|$. Since there are in A_1 not infinitely small elements, we have $\| u_{2k-1} \| \neq 0$. Hence $\| u_{2k-1} \|\ \| x \| \geqq 1$ for all k's. But this is in contradiction with the fact that $\lim \| u_n \| = 0$. Thus our theorem is proved.

If the ring A_1 is regular but not complete, the lattice of projections may contain an infinite set of mutually orthogonal equivalent projections. Take, for instance, the ring of all infinite matrices of complex numbers where all but a finite number of elements are zero. This ring is evidently *regular, satisfies the axiom (B), and all its elements are bounded. The lattice of projections of this ring is not a continuous geometry and it contains an infinite set of mutually orthogonal equivalent projections.

Though a complete *regular ring, of which all elements are bounded is always of type I, it is not necessarily a finite dimensional ring in the sense that there does not exist an infinite set of mutually orthogonal projections. Take, for instance, the ring of all sequences of complex numbers where each sequence is formed with a finite number of different complex numbers only. The ring operations are defined in the usual way. This ring is evidently a complete commutative *regular ring satisfying the axiom (B). All its elements are bounded. But it contains an infinite set of mutually orthogonal projections.

(Received 20.VIII.59)

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