ON THE QUADRATIC FUNCTIONAL

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A real-valued function $L(x)$ which is defined on a real Banach space R we call a quadratic functional if

$$L(x+y)+L(x-y)=2L(x)+2L(y) \tag{1}$$

holds true for every pair $x, y \in R$.

For the quadratic functional $L(x)$ we say that it is continuous in the point x_0 if $L(x_n)$ tends to $L(x_0)$ whenever x_n tends to x_0.

We say that the quadratic functional $L(x)$ is bounded on the sphere of radius r around x_0 if

$$\sup_{\|x\| \leqslant r} |L(x+x_0)| < +\infty.$$

The main result of this paper is that the continuity of a quadratic functional in one point or boundedness on one sphere implies the continuity everywhere and that a continuous quadratic functional $L(x)$ on a real Hilbert space has the form $L(x)=(Ax, x)$ where A is a symmetric and bounded transformation.

THEOREM 1. *Let $f(x)$ be a real-valued function which is defined on the set of real numbers and let*

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{2}$$

hold true for all real numbers x and y.

If the function $f(x)$ is bounded on the set of positive Lebesgue's measure, then

$$f(x)=x^2 f(1)$$

for every real number x.

The proof of this theorem is based on the following lemma.

LEMMA 1. *Let P be a perfect set of positive numbers and let the Lebesgue's measure of P be positive and finite.*

There exists a number $\varepsilon > 0$ with the property that for every $x \in [0, \varepsilon]$ a number y can be found such that

$$y,\ y+x/2,\ y+x \in P \qquad ([2],\ \text{lemma 1}).$$

Proof of theorem 1.

I. *For every real number x and for every rational number r we have:*

$$f(rx) = r^2 f(x). \tag{3}$$

Indeed, for $x=y=0$ (2) gives $f(0)=0$. For $x=0$ we get $f(-y)=f(y)$, i. e. $f(x)$ is an even function. Setting $x=y$ in (2) we find $f(2x)=2^2 f(x)$ and by the induction

$$f(nx) = n^2 f(x) \tag{4}$$

for every natural number n. Replacing x by x/n in (4) we find

$$f(x) = n^2 f(x/n).$$

Thus:

$$f\left(\frac{n}{m}x\right) = n^2 f\left(\frac{x}{m}\right) = \left(\frac{n}{m}\right)^2 f(x).$$

Since f is an even function (3) follows.

II. *The function $f(x)$ is bounded on one interval of the type $[0, a]$ $(a > 0)$.*

Since $f(x)$ is an even function which is bounded on the set of positive measure we can assume (without loss of generality) that $f(x)$ is bounded on the perfect set $P \subseteq [0, \infty)$ of positive and finite measure. Let

$$|f(x)| \leqslant A < +\infty$$

for every $x \in P$. Replacing x by $x+\ y$ in (2) we get:

$$\frac{1}{2}|f(2y)| = 2|f(y)| \leqslant 2|f(x+y)| + |f(x+2y)| + |f(x)|. \tag{5}$$

According to lemma 1 there exists a number $\varepsilon > 0$ such that for every $2y \in [0, \varepsilon]$ there exists the corresponding number x with the property that x, $x+2y/2$, $x+2y \in P$. If for the number $2y \in [0, \varepsilon]$ we take the corresponding x then (5) implies:

$$\frac{1}{2}|f(2y)| \leqslant 4A \quad \text{or} \quad |f(y)| \leqslant 2A,$$

i. e. $f(x)$ is bounded on the interval of the type $[0, a]$ $(a > 0)$.

III. Now we introduce the function (6)

$$h(x) = f(x) - x^2 f(1)$$

which is bounded on the interval $[0, r]$ (r is rational number and $0 < r \leqslant a$) and which satisfies the functional equation (2).

Let

$$B = \sup_{x \in [0, r]} |h(x)|.$$

If $B > 0$ then a number $x_0 \in [0, r]$ exists such that

$$|h(x_0)| > \frac{4}{5} B.$$

For this x_0 we have:

$$|h(2x_0)| = 4|h(x_0)| > 16B/5.$$

On the other hand the number $2x_0$ can be written in the form $2x_0 = r + y$ where y and $r - y$ are from the interval $[0, r]$. Using the functional equation (2) we have:

$$h(2x_0) = h(r+y) = 2h(y) + 2h(r) - h(r-y).$$

Since $h(r) = 0$ we find:

$$|h(2x_0)| \leqslant 2|h(y)| + |h(r-y)| \leqslant 3B.$$

Thus:

$$3B \geqslant |h(2x_0)| > 16B/5$$

which contradicts $B > 0$.

Thus $B = 0$ and $h(x) = 0$ on the interval $[0, r]$. For every real number y a rational number r' exists such that $r'y \in [0, r]$. We have: $r'^2 h(y) = h(r'y) = 0$ or $h(y) = 0$ for every real number y. This and (6) imply the assertion of theorem 1.

COROLLARY 1. *A measurable real-valued function of a real variable which satisfies* (2) *is continuous.*

Indeed, according to the well known Luzin's theorem ([4]), every measurable function is continuous on a perfect set of positive Lebesgue's measure. On the other hand a function which is continuous on a perfect and bounded set is bounded itself. This implies that a measurable function which satisfies (2) is bounded on a perfect and bounded set of positive measure. Theorem 1 implies the continuity of this function.

COROLLARY 2. *A real-valued function $g(x)$ of a real variable x which satisfies Cauchy's functional equation*

$$g(x+y) = g(x) + g(y)$$

and which is bounded on a set of positive Lebesgue's measure has the form

$$g(x) = x\, g(1), \qquad ([6], [1]).$$

This follows from the theorem 1 and the fact that $g^2(x)$ satisfies the functional equation (2).

If $H = \{H_\alpha\}$ is a Hamel's base of real numbers then every real number x can be written in the form

$$x = \sum_\alpha r_\alpha H_\alpha$$

where r_α are rational numbers and only a finite number of r_α's are different from zero. If we put:

$$f(x) = \sum r_\alpha^2 f(H_\alpha)$$

then f satisfies (2). From this follows the existence of a function $f(x)$ which satisfies (2) and which does not keep the sign on the set of real numbers. This function is not, therefore, the square of a solution of Cauchy's functional equation.

THEOREM 2. *Let $L(x)$ be a quadratic functional which is defined on a real Banach space R.*

If the quadratic functional $L(x)$ is continuous in one point $x_0 \in R$ then it is continuous everywhere and it is bounded on every finite sphere.

Conversely, if the quadratic functional $L(x)$ is bounded on one finite sphere then it is continuous.

For the proof of this theorem we need two lemmas.

LEMMA 2. *If the quadratic functional $L(x)$ is bounded on one finite sphere then it is bounded on every finite sphere.*

Proof: Suppose that:

$$\sup_{\|x\| \leqslant \varepsilon} |L(x + x_0)| < +\infty \tag{7}$$

where x_0 is an element around which $L(x)$ is bounded and $\varepsilon > 0$. Functional equation (1) gives

$$2L(x) = L(x + x_0) + L(x - x_0) - 2L(x_0)$$

which together with (7) implies:

$$\sup_{\|x\| \leqslant \varepsilon} |L(x)| < +\infty. \tag{8}$$

From (8) we conclude that:

$$\sup_{|t| \leqslant 1} |L(tx)| < +\infty \tag{9}$$

for every real number $t \in (-1, 1)$ and for every $x \in R$ the norm of which is less or equal to ε. On the other hand, for a given x, the function $L(tx)$ as a function of the real variable t satisfies the functional equation (2). Since it is bounded on one interval it is continuous (theorem 1) and

$$L(tx) = t^2 L(x) \tag{10}$$

for every real number t.

For an arbitrarily taken positive number A we have

$$\sup_{\|x\| \leqslant A} |L(x)| = \sup_{\|y\| \leqslant \varepsilon} \left| L\left(\frac{A}{\varepsilon} y\right) \right| = \left(\frac{A}{\varepsilon}\right)^2 \sup_{\|y\| \leqslant \varepsilon} |L(y)| < +\infty,$$

i.e. the functional $L(x)$ is bounded on every finite sphere.

LEMMA 3. *If the quadratic functional $L(x)$ is bounded on a finite sphere then it is continuous in zero and conversely.*

Proof. According to lemma 2 we have

$$\sup_{\|x\| \leqslant 1} |L(x)| < +\infty. \tag{11}$$

We assert that $x_n \to 0$ implies $L(x_n) \to L(0) = 0$. Otherwise we could find a number $a > 0$ and a sequence $x_n \to 0$ $(x_n \neq 0)$ such that

$$|L(x_n)| \geqslant a > 0 \tag{12}$$

for all n. As we have seen in the proof of lemma 2, (11) implies (10) for every real number t and for every $x \in R$. Thus:

$$\left| L\left(\frac{x_n}{\|x_n\|}\right) \right| = \frac{1}{\|x_n\|^2} |L(x_n)| \geqslant \frac{a}{\|x_n\|^2} > 0. \tag{13}$$

Since $x_n \to 0$ we have that $\|x_n\| \to 0$. This, (13) and the assumption $a > 0$ leads to the conclusion that the functional $L(x)$ is not bounded on the unit sphere, which contradicts (11). Therefore (11) implies the continuity of $L(x)$ in the origin. Thus the first part of lemma 3 is proved.

Now we assert that the continuity of $L(x)$ in zero implies its boundedness on the unit sphere. Otherwise we could find a sequence $x_n \in R$ such that:

$$x_n \to 0 \quad \text{and} \quad |L(x_n)| \geqslant n^2$$

for all natural numbers n. This gives:

$$|L(x_n/n)| \geqslant 1$$

which is impossible because x_n/n tends to zero and $L(x)$ is continuous in zero. In such a way lemma 3 is proved.

Proof of theorem 2. According to the assumption of theorem 2 the functional $L(x)$ is continuous in x_0, i. e.

$$L(x_0 \pm x_n) \to L(x_0)$$

whenever $x_n \to 0$. Functional equation (1) gives:

$$2L(x_n) = L(x_0 + x_n) + L(x_0 - x_n) - 2\ L(x_0) \to 0,$$

i. e. $x_n \to 0$ implies $L(x_n) \to 0$. Thus the functional $L(x)$ is also continuous in zero. We want to prove that $L(x)$ is contionuous in every point. Suppose that this is not so. There is, therefore, a real number $a > 0$, an element $z \in R$ and a sequence $x_n \in R$ such that:

$$| L(z + x_n) - L(z) | \geqslant a > 0 \tag{14}$$

and $x_n \to 0$. Since the functional $L(x)$ is continuous in zero it is bounded on every finite sphere (lemma 3). This implies:

$$| L(z + x) \leqslant L < +\infty \tag{15}$$

for all $\| x \| \leqslant 1$. From (15) and (14) we infer that the sequence of real numbers $L(z+x_n) - L(z)$ is bounded. Hence there is a subsequence x_{n_p} of the sequence x_n and a real number b such that:

$$\lim_{n_p \to \infty} [L(z + x_{n_p}) - L(z)] = b. \tag{16}$$

From (16) and (14) we conclude that $b \neq 0$. Since b is not zero there is a natural number k such that:

$$| b | > \frac{2L + 1}{2^k}. \tag{17}$$

As x_{n_p} tends to zero there is only a finite number of x_{n_p}'s for which the relation

$$\| x_{n_p} \| \leqslant 1/2^k \tag{18}$$

does not hold. We take into account only those x_{n_p} for which (18) is valid and the so obtained sequence we again denote by x_n. For the sequence x_n (14) holds and

$$\| x_n \| \leqslant 1/2^k, \qquad \lim_{n \to \infty} L(z + x_n) = b + L(z). \tag{19}$$

Setting $x+y$ instead of x in (1) we get:

$$L(x+2y)=2L(x+y)+2L(y)-L(x). \tag{20}$$

If in (20), we put $x=z$ and $y=x_n$ then we get:

$$L(z+2x_n)=2L(z+x_n)+2L(x_n)-L(z).$$

From here and (19) we find:

$$\lim_{n\to\infty} L(z+2x_n)=2b+L(z). \tag{21}$$

If we put $x=z$, $y=2x_n$ in (20) we find:

$$L(z+4x_n)=2L(z+2x_n)+2L(2x_n)-L(z)$$

which together with (21) leads to:

$$\lim_{n\to\infty} L(z+2^2x_n)=2^2b+L(z)$$

and by induction:

$$\lim_{n\to\infty} L(z+2^kx_n)=2^kb+L(z). \tag{22}$$

From (22) we deduce the existence of a natural number n_0 such that:

$$|L(z+2^kx_n)-L(z)|>2^k|b|-1 \tag{23}$$

for all $n\geqslant n_0$. On the other hand (19) implies $\|2^2 x_n\|\leqslant 1$. This together with (15) and (23) implies:

$$2L>2^k|b|-1$$

which contradicts (17). Therefore the assumption that $L(x)$ is not continuous in one point leads to the contradiction. Thus the functional $L(x)$ is continuous everywhere.

The second part of theorem 2 follows from lemma 3 and the first part of theorem 2. In such a way theorem 2 is completely proved.

THEOREM 3. *Let $L(x)$ be a quadratic functional which is defined on a real Hilbert space R.*

If the quadratic functional $L(x)$ is continuous then there exists one and only one symmetric and bounded linear transformation A such that:

$$L(x)=(Ax,x) \tag{24}$$

for all $x\in R$.

The proof of this theorem is carried on in several steps.

I. *Uniqueness.* If A and B are symmetric transformations and if

$$(Ax, x) = (Bx, x) \tag{25}$$

for all $x \in R$ then $A = B$. Indeed (25) implies:

$$(A(x+y), x+y) - (Ax, x) - (Ay, y) = (B(x+y), x+y) - (Bx, x) - (By, y)$$

from which one obtains:

$$(Ax, y) = (Bx, y)$$

for all $x, y \in R$. Thus $A = B$.

II. *The space R is n-dimensional.* If we take an orthonormal basic set $e_1, e_2, \ldots, e_n$ in R then we have:

$$L(x) = L\left(\sum_{k=1}^{n} x_k e_k\right) = F(x_1, x_2, \ldots, x_n) \tag{26}$$

and the function F satisfies the following functional equation:

$$F(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) + F(x_1-y_1, x_2-y_2, \ldots, x_n-y_n) =$$
$$= 2F(x_1, x_2, \ldots, x_n) + 2F(y_1, y_2, \ldots, y_n). \tag{27}$$

According to the theorem 2 the function F is a continuous function. If we integrate (27) from a to b with respect to x_1 then we get:

$$\begin{aligned} 2(b-a)F(y_1, y_2, \ldots, y_n) + 2\int_a^b F(x_1, x_2, \ldots, x_n)dx_1 = \\ = \int_{a+y_1}^{b+y_1} F(u, x_2+y_2, \ldots, x_n+y_n)\,du + \int_{a-y_1}^{b-y_1} F(u, x_2-y_2, \ldots, x_n-y_n)\,du. \end{aligned} \tag{28}$$

From (28) we conclude that

$$\frac{\partial}{\partial y_1} F(y_1, y_2, \ldots, y_n)$$

exists and that this derivative is expressed as a linear combination of functions:

$$F(b \pm y_1, x_2+y_2, \ldots, x_n+y_n), \quad F(a \pm y_1, x_2+y_2, \ldots, x_n+y_n). \tag{29}$$

We have just proved that every function in (29) has the partial derivative with respect to the first variable. This proves that

$$\frac{\partial^2}{\partial y_1^2} F(y_1, y_2, \ldots, y_n)$$

exists and that it is a continuous function. In the similar way we see that

$$\frac{\partial^2}{\partial x_k} F(x_1, x_2, \ldots x_n) \qquad (k=1, 2, \ldots, n) \tag{30}$$

exists and that it is a continuous function.

If we take the second partial derivative of (27) with respect to x_1 we find:

$$\frac{\partial^2}{\partial y_1^2} F(x_1+y_1, x_2+y_2, \ldots, x_n+y_n) + \frac{\partial^2}{\partial y_1^2} F(y_1-x_1, y_2-x_2, \ldots, y_n-x_n) = $$
$$= 2\frac{\partial^2}{\partial x_1^2} F(x_1, x_2, \ldots, x_n). \tag{31}$$

Setting $x_1=x_2= \ldots =x_n=0$ in (31) we get:

$$\frac{\partial^2}{\partial y_1^2} F(y_1, y_2, \ldots, y_n) = \left[\frac{\partial^2}{\partial x_1^2} F(x_1, x_2, \ldots, x_n)\right]_{x=0} = 2a_{11} \tag{32}$$

where $2a_{11}$ designates a number.

Integrating (32) we obtain:

$$F(x_1, x_2, \ldots, x_n) = a_{11} x_1^2 + x_1 G(x_2, x_3, \ldots, x_n) - F_1(x_2, x_3, \ldots, x_n) \tag{33}$$

If we put $x_1=0$ then we find that F_1 is a continuous function of $n-1$ variables which satisfies the functional equation (27). Since $a_{11} x_1^2$ satisfies (27) we conclude that the function

$$x_1 G(x_2, x_3, \ldots, x_n) \tag{34}$$

satisfies the functional equation (27). Setting (34) in (27) and taking $x_1=y_1\neq 0$ we obtain:

$$G(x_2+y_2, x_3+y_3, \ldots, x_n+y_n) = G(x_2, x_3, \ldots, x_n) + G(y_2, y_3, \ldots, y_n). \tag{35}$$

Since the function G is continuous we get from (35)

$$G(x_2, x_3, \ldots, x_n) = 2a_{12}x_2 + 2a_{13}x_3 + \ldots + 2a_{1n}x_n. \tag{36}$$

Now (36) and (33) lead to:

$$F(x_1, x_2, \ldots, x_n) = a_{11}x_1^2 + 2\sum_{k=2}^{n} a_{1k}x_1x_k + F_1(x_2, x_3, \ldots, x_n). \tag{37}$$

Clearly the same method implies:

$$F(x_1, x_2, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij}\, x_i\, x_j, \quad a_{ij} = a_{ji}. \tag{38}$$

In such a way in the basic set $e_1, e_2, \ldots, e_n$ we have:

$$L\left(\sum_{k=1}^{n} x_k\, e_k\right) = \sum_{i,j=1}^{n} a_{ij}\, x_i\, x_j. \tag{39}$$

Let A be the linear transformation in R which has in the basic set $e_1, e_2, \ldots, e_n$ the matrix (a_{ij}). Obviously A is a symmetric transformation. If we take another orthonormal basic set $e_1', e_2', \ldots, e_n'$ then an orthogonal transformation O exists such that $e_k' = Oe_k$. We have:

$$L(x) = L\left(\sum_{i=1}^{n} x_i'\, e_i'\right) = \sum_{i,j=1}^{n} a_{ij}'\, x_i'\, x_j'.$$

and

$$L(x) = L\left(\sum_{i=1}^{n} x_i'\, Oe_i\right) = L\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{n} O_{ji}\, x_i'\right) e_j\right] =$$

$$= \sum_{p,q=1}^{n} a_{pq}\left(\sum_{i=1}^{n} O_{pi}\, x_i'\right)\left(\sum_{i=1}^{n} O_{qi}\, x_i'\right).$$

Thus:

$$a_{ij}' = \sum_{p,q=1}^{n} O_{ip}'\, a_{pq}\, O_{qj}$$

which proves that the matrix (a_{ip}') represents in the basic set $e_1', e_2', \ldots, e_n'$ the same linear transformation which the matrix (a_{ij}) represents in the basic set $e_1, e_2, \ldots, e_n$, i. e. (a_{ij}') is a matrix of A in the basic set $e_1', e_2', \ldots, e_n'$. This conclusion and (39) imply

$$L(x) = (Ax, x) \tag{40}$$

for every $x \in R$. This ends the proof of theorem 3 in the case of a finite dimensional space.

III. *The space R is a Hilbert space (separable or not).* We define the functional $M(x, y)$ by the relation:

$$M(x, y) = \frac{1}{2}\left[L(x+y) - L(x) - L(y)\right]. \tag{41}$$

As we see the functional $M(x, y)$ is symmetric and since $L(x)$ is bounded we find:

$$\sup_{\|x\| \leq 1, \|y\| \leq 1} |M(x, y)| < +\infty, \tag{42}$$

i.e. $M(x, y)$ is a bounded symmetrical functional. We want to prove that $M(x, y)$ is a bilinear functional. For this it is sufficient to prove that:

$$M(ax'+by', z') = a\,M(x', z') + b\,M(y', z') \tag{43}$$

for every pair of real numbers a, b and for any elements $x', y', z' \in R$. In order to prove (43) for given (but arbitrary) x', y' and z' we consider the three dimensional subspace R_3 of R which contains the vectors x', y' and z'. Let e_1, e_2, e_3 be an orthonormal basic set in R_3. Then any vector $x \in R_3$, has the form

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3 .$$

If we consider the functional (1) only for $x, y \in R_3$, then we conclude that $L(x)$ is a quadratic functional on R_3 $(x \in R_3)$. This and II imply

$$L(x) = L\left(\sum_{k=1}^{3} x_k e_k\right) = \sum_{i,j=1}^{3} a_{ij} x_i x_j \tag{44}$$

for every $x \in R_3$. Thus:

$$M(x, y) = \sum_{i,j=1}^{3} a_{ij} x_i y_j \tag{45}$$

for any $x, y \in R_3$. Since x', y' and z' are elements of R_3, (45) leads to:

$$M(ax'+by', z') = \sum_{i,j=1}^{3} a_{ij} (ax'+by')_i z_j' =$$

$$= a \sum_{i,j=1}^{3} a_{ij} x_i' z_j' + b \sum_{i,j=1}^{3} a_{ij} y_i' z_j' =$$

$$= a\,M(x', z') + b\,M(y', z'),$$

i. e. (43) holds.

Since $M(x, y)$ is a bounded bilinear and symmetrical functional we have ([5] pp. 86—87)

$$M(x, y) = (Ax, y) \tag{46}$$

for any couple $x, y \in R$ where A is a bounded and symmetrical transformation. Now (46) and (41) imply:

$$L(x) = M(x, x) = (Ax, x)$$

for every $x \in R$.

Q. E. D.

COROLLARY 3. *Let the quadratic functional* $L(x)$ *satisfy all conditions of theorem* 3.

If

$$L(Ox)=L(x) \tag{47}$$

for every $x \in R$ *and for every orthogonal transformation* O, *then*

$$L(x)=a\|x\|^2 \tag{48}$$

for every $x \in R$ *where* a *is a real number.*

Proof. According to theorem 3 we have $L(x)=(Ax,x)$ for every $x \in R$. This and (47) imply:

$$(Ax,x)=(OAx,Ox)=(O'AOx,x)$$

or by the fact that $O'AO$ is a symmetric transformation we get:

$$O'AO=A$$

for every orthogonal transformation O, which is possible only if A is a multiple of the identity transformation.

If $L(e)=1$ for some vector $e \in R$, $\|e\|=1$, then

$$L(x)=\|x\|^2$$

for any $x \in R$.

THEOREM 4. *Let to every real square matrix* x *of order* n *a real number* $L(x)$ *be attached in such a way that:*

$$L(x+y)+L(x-y)=2L(x)+2L(y)$$

hold for all such matrices x, y *and let:*

$$L(s^{-1}xs)=L(x)$$

for every matrix x *and for every non singular real matrix* s.

If $L(x)$ *is a continuous functional then:*

$$L(x)=a\left(\sum_{i=1}^{n} x_{ii}\right)^2+b\sum_{1\leqslant i<j\leqslant n}\begin{vmatrix} x_{ii} & x_{ij} \\ x_{ji} & x_{jj}\end{vmatrix} \tag{49}$$

for all x, *where* a *and* b *are real numbers which do not depend on* x.

The proof of this theorem is based on the theorem 3 and it is given in ([3], theorem 9).

THEOREM 5. *Let a and b be real numbers; let $L(x)$, $F(x)$ and $G(x)$ be real-valued functionals which are defined on a real Hilbert space R and let*

$$L(x+y)+a\,L(x-y)=2\,F(x)+2\,G(y)+2\,b \tag{50}$$

hold for any couple $x, y \in R$.

If for some $\varepsilon>0$

$$\sup_{\|x\|\leq\varepsilon}|L(x)|<+\infty \tag{51}$$

then

$$L(x)=\frac{2}{1+a}(Ax,x)+(x,x_0)+L(0),$$

$$G(x)=(Ax,x)+\frac{1-a}{2}(x,x_0)+G(0)$$

and

$$F(x)=(Ax,x)+\frac{1+a}{2}(x,x_0)+\frac{1+a}{2}L(0)-G(0)-b,$$

for all $x \in R$, where A is a bounded symmetric transformation in R and x_0 is a vector of R. The vector x_0 and the symmetric transformation A are uniqually determined by the functional $L(x)$.

In the case $a \neq 1$ in all formulae we have to put $A=A/1+a=0$.

Proof. For $y=0$ (50) gives:

$$F(x)=\frac{1+a}{2}L(x)-G(0)-b. \tag{52}$$

Setting (52) in (50) we get:

$$L(x+y)+a\,L(x-y)=(1+a)\,L(x)+2\,g(y) \tag{53}$$

where

$$g(y)=G(y)-G(0). \tag{54}$$

Denote:

$$\begin{aligned} L(x)+L(-x)&=2\,K(x), & g(x)+g(-x)&=2\,N(x),\\ L(x)-L(-x)&=2\,l(x), & g(x)-g(-x)&=2\,M(x). \end{aligned} \tag{55}$$

If we replace x, y in (53) by $-x$ and $-y$ then, using (55) and (53) we get:

$$K(x+y)+a\,K(x-y)=(1+a)\,K(x)+2\,N(y), \tag{56}$$

$$l(x+y)+a\,l(x-y)=(1+a)\,l(x)+2\,M(y). \tag{57}$$

If we interchange x and y in (57) and if we add this result to (57) we find:

$$l(x+y)=\frac{1+a}{2}[l(x)+l(y)]+M(x)+M(y). \qquad (58)$$

For $y=0$ (58) leads to:

$$(1-a)\,l(x)=2M(x). \qquad (59)$$

By use of (59), (58) reads:

$$l(x+y)=l(x)+l(y). \qquad (60)$$

Now (60) and (51) imply that $l(x)$ is a bounded and additive functional. There exists, therefore, a unique vector $x_0 \in R$ such that:

$$l(x)=(x, x_0) \qquad (61)$$

for all $x \in R$ ([5], p. 86). This gives:

$$M(x)=(1-a)\ (x, x_0)/2. \qquad (62)$$

Replacing y by $-y$ in (56) we get:

$$K(x-y)+aK(y+x)=(1+a)K(y)+2N(x). \qquad (63)$$

If we add (63) and (56) we obtain:

$$(1+a)[K(x+y)+K(x-y)]=(1+a)[K(x)+K(y)]+2N(x)+2N(y). \quad (64)$$

For $x=0$ (56) gives:

$$(1+a)K(y)=(1+a)K(0)+2N(y). \qquad (65)$$

Now (65) and (64) lead to:

$$N(x+y)+N(x-y)=2N(x)+2N(y). \qquad (66)$$

According to (55) and (51) the functional $K(x)$ is bounded. This and (65) imply that the functional $N(x)$ is bounded too. Theorem 3 implies the existence of a symmetric transformation A such that:

$$N(x) = (Ax, x) \tag{67}$$

for every $x \in R$.

We have two cases to discuss.

I. $1+a \neq 0$ and II. $1+a=0$.

In the case $1+a \neq 0$, (65) and (67) give:

$$K(x) = \frac{2}{1+a} \cdot (Ax, x) + L(0), \tag{68}$$

because $L(0) = K(0)$. Now (68), (62), (61), (54), (52) and (55) lead to the expressions for $L(x)$, $F(x)$ and $G(x)$ which we gave in theorem 5. Replacing these expressions in (50) we find $A=0$ for $a \neq 1$.

In the case $a=-1$, (65) leads to $N(x)=0$ for every $x \in R$. Thus $A=0$. Since $N(x)=0$ we have from (55) that $g(x)$ is an odd function. This and (62) give:

$$g(x) = M(x) = \frac{1-a}{2}(x, x_0) = (x, x_0). \tag{69}$$

If we put (69) in (53) we get:

$$L(x+y) - L(x-y) = 2(y, x_0)$$

which implies:

$$L(2x) = 2(x, x_0) + L(0) = (2x, x_0) + L(0)$$

or

$$L(x) = (x, x_0) + L(0) \tag{70}$$

for every $x \in R$. Now (69), (54) and (52) imply:

$$G(x) = G(0) + (x, x_0) \quad \text{and} \quad F(x) = -G(0) - b.$$

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