

ON THE PRODUCT $\prod_{k=1}^n (1 - z^{a_k})$.

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Consider the product $\prod_{k=1}^n (1 - z^{a_k})$ where $a_1 \leq a_2 \leq \dots \leq a_n$ are positive integers. Put

$$\max_{|z|=1} \prod_{i=1}^n (1 - z^{a_i}) = M(a_1, a_2, \dots, a_n), \quad f(n) = \min_{a_1, a_2, \dots, a_n} M(a_1, a_2, \dots, a_n).$$

Clearly $M(a_1, \dots, a_n) \leq 2^n$ (equality if and only if $(a_1, a_2, \dots, a_n) > 1$ or $a_1 = a_2 = \dots = a_n = 1$). The determination of $f(n)$ seems to be a very difficult question, and even a good estimation of $f(n)$ does not seem easy. In the present note we are going to prove that $f(n)^{1/n} \rightarrow 1$ as $n \rightarrow \infty$, and it seems possible that a refinement of our method would give $(\exp z = e^z)$

$$f(n) < \exp(n^{1-c})$$

for some $c < 1$. The lower bound $f(n) \geq \sqrt{2n}$ is nearly trivial, and we are unable at present to do any better.

We want to remark that it is easy to show that

$$\lim_{n \rightarrow \infty} [M(1, 2, \dots, n)]^{1/n}$$

exists and is between 1 and 2.

Put $z = e^{2\pi i \alpha}$, $\langle \alpha \rangle = |1 - e^{2\pi i \alpha}|$. Several further questions can be asked. It is not difficult to prove that for almost all α (almost all means except a set of Lebesgue measure 0)

$$(1) \quad \lim_{n \rightarrow \infty} \prod_{k=1}^n \langle k\alpha \rangle = 0.$$

We only outline the proof of (1). A special case of a well known theorem of Khintchine states that for almost all α there is an infinite sequence of integers p_n and q_n satisfying

$$(2) \quad \left| \alpha - \frac{p_n}{q_n} \right| = o\left(\frac{1}{q_n^2 \log q_n}\right).$$

A simple computation then shows that

$$\lim_{k \rightarrow \infty} \prod_{n=1}^{q_n} \langle k\alpha \rangle = 0.$$

Perhaps (1) holds for all α .

It is easy to see that

$$(3) \quad \overline{\lim} \prod_{k=1}^n \langle k\alpha \rangle = \infty.$$

holds for almost all α . (Clearly (3) can not hold for all α , e. g. it fails if α is rational). To see this we observe that a simple computation shows that if the q_n are the integers satisfying (2) then

$$\lim_{k \rightarrow \infty} \prod_{n=1}^{q_n-1} \langle k\alpha \rangle = \infty.$$

Perhaps one could determine how fast (1) tends to 0 and (3) tends to ∞ for almost all α .

Is it true that for all α

$$(4) \quad \overline{\lim} \max_{|z|=1} \prod_{k=1}^n |z - e^{2\pi i k \alpha}| = \infty?$$

An old conjecture of P. Erdős which would imply (4) states as follows: Let z_1, z_2, \dots be any infinite sequence satisfying $|z_i| = 1$.

Then

$$\overline{\lim} \max_{|z|=1} \prod_{i=1}^n |z - z_i| = \infty.$$

On the other hand a simple computation shows that for the q_n satisfying (2)

$$(5) \quad \lim_{k \rightarrow \infty} \max_{|z|=1} \prod_{k=1}^{q_n} |z - e^{2\pi i k \alpha}| = 2,$$

and perhaps

$$\overline{\lim} \max_{|z|=1} \prod_{k=1}^n |z - e^{2\pi i k \alpha}| < \infty$$

for all irrational α (it certainly is ∞ for rational α).

On the product $\prod_{k=1}^n (1 - z^{a_k})$

Finally we pose the following problem: Let $a_1 < a_2 < \dots$ be any infinite sequence of integers. Is it true that for almost all α

$$\overline{\lim} \prod_{k=1}^n \langle a_k \alpha \rangle = \infty, \quad \underline{\lim} \prod_{k=1}^n \langle a_k \alpha \rangle = 0?$$

Throughout this paper $0 \leq \alpha < 1$ and c_1, c_2, \dots will denote positive absolute constants, $|\theta| < 1$ (and the θ 's appearing are not necessarily the same).

LEMMA 1. *Let*

$$(6) \quad \alpha = \frac{p}{q} + \frac{\theta}{q^2}, \quad (p, q) = 1.$$

Then for every l

$$(7) \quad \prod_{t=l+1}^{l+q} \langle t\alpha \rangle < q^{a_1}.$$

If $\theta = 0$ the product in (7) is 0, hence (7) holds. Thus we can assume $\theta \neq 0$. Order the numbers $e^{2\pi i t \alpha}$, $l+1 \leq t \leq l+q$ according to the size of their arguments and denote them by z_1, z_2, \dots, z_q ($0 < \arg z_1 < \arg z_2 < \dots < \arg z_q < 2\pi$, i. e. (6) implies that the z 's are all different). From (6) we have

$$(8) \quad \arg z_k = 2\pi \left(\frac{k}{q} + \frac{\theta}{q} \right) \quad (k = 1, 2, \dots, q).$$

Put $y_k = e^{2\pi i (k-1/2)/q}$. From (8) we evidently have

$$(9) \quad |1 - z_k| < |1 + y_k| \left(1 + c_2 \left(\frac{1}{k} + \frac{1}{q+1-k} \right) \right).$$

Now from $\prod_{k=1}^q |1 - y_k| = 2$, $\left(\prod_{k=1}^q |1 - y_k| \right)$ is simply the value at $z = 1$ of $(z^{2q} - 1)/(z^k - 1) = z^k + 1$) and (9) we have

$$\prod_{k=1}^q |1 - z_k| < 2 \prod_{k=1}^q \left(1 + c_2 \left(\frac{1}{k} + \frac{1}{q+1-k} \right) \right) < q^{c_1},$$

which proves the Lemma.

THEOREM 1. *To every ϵ there exists an $n_0(\epsilon)$ and $A = A(\epsilon)$, $B = B(\epsilon)$ so that for every $n > n_0(\epsilon)$ and every α which does not satisfy one of the*

inequalities

$$(10) \quad \frac{1}{Bn} < \left| \alpha - \frac{p}{q} \right| \leq \frac{1}{\varepsilon n} \quad \text{for some } 0 \leq p < q \leq A$$

we have

$$(11) \quad \prod_{t=1}^n \langle t\alpha \rangle < (1+\varepsilon)^n.$$

Theorem 1 means that (11) is satisfied except if α can be approximated „well“ but not „too well“ by rational fractions with „small“ denominators.

Assume first that α is such that for every p and $q \leq A$

$$(12) \quad \left| \alpha - \frac{p}{q} \right| \geq \frac{1}{\varepsilon n}.$$

By a well known theorem of Dirichlet there exists a $q \leq \varepsilon n$ for which

$$(13) \quad \left| \alpha - \frac{p}{q} \right| < \frac{1}{q\varepsilon n} < \frac{\theta}{q^2} \quad (p, q) = 1.$$

By (12) $q > A$. Put $uq \leq n < (u+1)q$. Then we have by $\langle t\alpha \rangle \leq 2$ and our Lemma (since $2^\varepsilon < 1+\varepsilon$ for small ε and $q^{1/q} < A^{1/A}$ for $q > A > e$)

$$(14) \quad \prod_{t=1}^n \langle t\alpha \rangle < 2^q q^{c_1 u} < 2^{\varepsilon n} q^{c_1 \frac{n}{q}} < 2^{\varepsilon n} A^{c_1 \frac{n}{A}} < (1+\varepsilon)^n$$

if $A > A(\varepsilon)$ is large enough.

If for some $q \leq A$, $\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{\varepsilon n}$ then by (10) we can assume that $\left| \alpha - \frac{p}{q} \right| < \frac{1}{Bn}$. But then the arguments of the numbers $e^{2\pi i(vq+l)\alpha}$, $v < u$, $1 \leq l \leq q$ ($uq \leq n < (u+1)q$) differ from the corresponding q -th roots of unity by less than $1/B$. Thus for B sufficiently large a simple computation gives

$$\prod_{vq < l \leq (v+1)q} \langle t\alpha \rangle < \frac{1}{2},$$

or

$$(15) \quad \prod_{t=1}^n \langle t\alpha \rangle < \prod_{t=1}^{uq} \langle t\alpha \rangle \prod_{uq < t \leq n} \langle t\alpha \rangle < \left(\frac{1}{2}\right)^u 2^q < \left(\frac{1}{2}\right)^{n/A} 2^A < 1$$

and Theorem 1, follows from (14) and (15).

On the product $\prod_{k=1}^n (1 - z^{a_k})$

Next we prove

THEOREM 2.

$$\lim f(n)^{1/n} = 1.$$

Let $m^2 \leq n < (m+1)^2$. Consider the product

$$g_n(z) = \prod_{k=1}^m \prod_{l=1}^m (1 - z^{2^k l}) (1 - z)^{n-m^2}.$$

In other words $a_1 = a_2 = \dots = a_{n-m^2} = 1$ and the other a 's are the integers $2^k l$, $1 \leq k \leq m$, $1 \leq l \leq m$. To prove Theorem 2 it will be sufficient to show that

$$(16) \quad \lim_{|z|=1} \max |g_n(z)|^{1/n} = 1.$$

We evidently have for $|z| \leq 1$

$$|1 - z|^{n-m^2} \leq 2^{2\sqrt{n}} \quad (\text{i. e. } n - m^2 \leq 2\sqrt{n}).$$

Thus to prove (16) it will suffice to show that for every ϵ if $m > m_0(\epsilon)$

$$(17) \quad \max_{|z|=1} \prod_{k=1}^m \prod_{l=1}^m |1 - z^{2^k l}| = \max_{0 \leq \alpha \leq 1} \prod_{k=1}^m \prod_{l=1}^m \langle 2^k l \alpha \rangle < (1 + 2\epsilon)^{m^2}.$$

Consider the numbers $2^k \alpha = \alpha_k$, $1 \leq k \leq m$. We claim that only $o(m)$ of them satisfy (10). Since $q \leq A$ it will suffice to show that only $o(m)$ of them satisfy (10) for a fixed q .

Suppose in fact that α_k satisfies (10) for a certain q . Then we have $\left| \alpha_k - \frac{p}{q} \right| = \frac{b_k}{n}$ where $\frac{1}{B} \leq |b_k| \leq \frac{1}{\epsilon}$. Also $\left| \alpha_{k+1} - \frac{p'}{q} \right| = \frac{2b_k}{n}$ where $p' \equiv 2p \pmod{q}$. Thus (10) can be satisfied for at most $\frac{\log B/\epsilon}{\log 2} + 1$ consecutive values of k and these are followed by at least $c_3 \log n$ values of k for which (10) is not satisfied for this particular value of q applying this argument for all the $k \leq m$ which satisfy (10) we obtain that (10) is satisfied for only $o(m)$ values of k , as stated.

Now we can prove (17). Write

$$\prod_{k=1}^m \prod_{l=1}^m \langle 2^k l \alpha \rangle = \prod_1 \prod_{k=1}^m \langle 2^k l \alpha \rangle \prod_2 \prod_{k=1}^m \langle 2^k l \alpha \rangle$$

where in Π_1 k is such that $2^k \alpha = \alpha_k$ satisfies (10). Clearly $\prod_{l=1}^m \langle 2^k / \alpha \rangle \leq 2^m$, thus by what we just proved

$$(18) \quad \prod_k \prod_{l=1}^m \langle 2^k / \alpha \rangle = 2^{o(m^2)}.$$

By Theorem 1 we have for every k in Π_2

$$\prod_{l=1}^m \langle 2^k / \alpha \rangle < (1 + \varepsilon)^m.$$

Thus

$$(19) \quad \prod_k \prod_{l=1}^m \langle 2^k / \alpha \rangle < (1 + \varepsilon)^{m^2}.$$

(18) and (19) implies (17), and thus Theorem 2 is proved.

THEOREM 3.

$$f(n) \geq \sqrt{2n}.$$

To prove Theorem 3 write

$$\prod_{i=1}^n (1 - x^{a_i}) = \sum_i x^{b_i} - \sum_i x^{c_i}, \quad b_1 < b_2 < \dots; c_1 < c_2 < \dots$$

First we show that

$$(20) \quad \sum_i b_i^p = \sum_i c_i^p, \quad p = 0, 1, \dots, n-1.$$

To show (20) observe that 1 is an n -fold root of $\prod_{i=1}^n (1 - x^{a_i})$. Thus $f^{(p)}(1) = 0$ for $p = 0, 1, \dots, n-1$; or

$$\sum_i b_i (b_i - 1) \dots (b_i - p + 1) = \sum_i c_i (c_i - 1) \dots (c_i - p + 1), \quad p = 0, 1, \dots, n-1,$$

which implies (20). From (20) we immediately obtain that at least n b 's and n c 's do not vanish which implies Theorem 3 by Parseval's equality.

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