# ON A CLASS OF FOURIER INTEGRALS

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**0.1.** Let $a_\nu (\nu = 1, 2, \ldots, 2m-1)$ denote arbitrary complex numbers $a_0 = 1$, and

$$P(x) = \sum_{\nu=0}^{2m-1} a_\nu x^{2m-\nu} \qquad \text{with } m \geqslant 2.$$

We consider the integral function $F(z)$, defined by means of a Fourier integral as follows:

$$F(z) = \int_{-\infty}^{\infty} e^{-P(t)+zt}\, dt. \tag{0.1}$$

and answer two questions of different kinds.

Firstly, we give the asymptotic expansion of $F(z)$ for $z \to \infty$, which problem is connected — more or less — with the investigation of many authors e. g. [1], [2], [4], [5], [8], [9].

Secondly, we show that $F(z)$ with $z = P'(y)$ can be expressed as a product of a function of an exponential type and a convergent series in Wright's hypergeometric functions.

For the proof we consider

$$J(y) \equiv F\{P'(y)\}$$

and show that the both results mentioned above simply follow from an inequality satisfied by $J(y)$ which is proved in § 1.2 by the saddle-point method.

**1.1.** Let $C_0, C_1, \ldots,$ stand for positive constants depending only on $m$ and a however small number $\varepsilon$ if otherwise is not explicitely stated. Further let $f(x; y) = \sum_{\nu=3}^{2m-1} \frac{P^{(\nu)}(y)\, y^\nu}{\nu!} x^\nu$. If $i = 3p,\ 3p+1, \ldots,\ p = 0, 1, \ldots, 2n$

then we define the polynomials $Q_{i,p}(y)$ by means of the formula

$$(1.1)\qquad \{f(x;y)\}^p=\sum_{i=3p}^{(2m-1)p} Q_{i,p}(y)x^i \text{ and } Q_{i,p}(y)\equiv 0,\ i>2m-1.$$

Further, for sake of brevity, we shall write

$$A_i(y)=\sum_{p=1}^{[i/3]}\frac{(-1)^p}{p!}Q_{i,p}(y) \text{ if } i\geqslant 3,\quad A_0(y)=1,\quad A_1(y)=A_2(y)\equiv 0$$

and

$$(1.2)\qquad \Omega_{2k}(y)=2y\int_{L_\varphi}\exp\left\{-\frac{P''(y)y^2}{2}x^2-x^{2m}y^{2m}\right\}x^{2k}\,dx,\quad k=0,1,\ldots$$

$$-\varphi-\frac{\pi}{4m}<\arg y<-\varphi+\frac{\pi}{2m}$$

where $L_\varphi$ is an infinite ray making an angle $\varphi$ with the positive part of the real axis[1].

**1.2.** We, now, prove the following

THEOREM 1. *If we put*

$$J(y)=e^{-P(y)+yP'(y)}\left\{\sum_{k=0}^{(2m-1)n}A_{2k}(y)\,\Omega_{2k}(y)+\tilde{R}_n(y)\right\}$$

*then there exist fixed positive numbers* $\alpha$, $\beta$ *so that the inequalities*

$$|\tilde{R}_n(y)|<\begin{cases}C_1\,|y|^{-m(2n+2)+1}\,n^{\alpha n}\\ C_2\,|y|^{2m(2n+1)+1}\,n^{-\beta n}\end{cases}$$

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[1] We can express the function $\Omega_{2k}$ by means of the Wright's generalisation of the hypergeometric function:

$${}_pF_q(t)=\sum_{n=0}^{\infty}\frac{f(n)}{\Gamma(n+1)}t^n$$

where

$$f(x)=\prod_{\nu=1}^{p}\Gamma(\beta_\nu+\alpha_\nu x)\left\{\prod_{\nu=1}^{q}\Gamma(\mu_\nu+\rho_\nu x)\right\}^{-1};$$

$\alpha_1,\alpha_2,\ldots,\rho_1,\rho_2,\ldots,$ are real numbers and $k=1+\rho_1+\cdots\rho_q-\alpha_1-\cdots-\alpha_p>0$. Viz it is easily verified that

$$\Omega_{2k}(y)=\frac{1}{my^{2k}}\,{}_1F_0\left\{-\frac{p''(y)}{2}\right\}\quad\text{with } \alpha_1=\frac{1}{m},\quad \beta_1=\frac{2k+1}{2n}.$$

*hold for every positive integer n, and all y belonging to the domains*

$$D_1: \qquad \varepsilon-\frac{\pi}{4m}\,\frac{m+1}{m-1}\leqslant \arg y\leqslant \frac{\pi}{4m}\,\frac{m+1}{m-1}-\varepsilon,$$

$$|y|>y_0$$

$$D_2: \qquad \varepsilon+\pi-\frac{\pi}{4m}\,\frac{m+1}{m-1}\leqslant \arg y\leqslant \pi+\frac{\pi}{4m}\,\frac{m+1}{m-1}-\varepsilon,$$

*where* $y_0$ *is a sufficiently large and* $\varepsilon$ *a however small number.*

For proof the following simple lemma is needed:

LEMMA 1. *Let* $\arg y=\theta$, $\arg x=\varphi$ *and*

$$(1.3) \qquad z_1(x;y)=\sum_{\nu=2}^{2m}\frac{P^{(\nu)}(y)\,y^\nu}{\nu!}\,x^\nu, \quad z_2(x;y)=\frac{P''(y)\,y^2}{2}\,x^2+y^{2m}x^{2m}$$

*then the inequalities*

$$\Re\{z_1\},\ \Re\{z_2\}>C_0|y|^{2m}\operatorname{Max}\{|x|^2,\ |x|^{2m}\}$$

*hold when at the same time*

$$(1.4) \qquad \varepsilon-\frac{\pi}{2}\leqslant 2m(\theta+\varphi)\leqslant\frac{\pi}{2}+\varepsilon, \quad \varepsilon-\frac{\pi}{2}\leqslant 2m\theta+2\varphi\leqslant\frac{\pi}{2}-\varepsilon$$

*for all* $|x|>0$ *and* $|y|>y_0$, $\varepsilon$ *being however small.*

Proof of the lemma 1. We rewrite the function $z_1(x;y)$ in the more convenient form

$$(1.5) \qquad z_1\{x;y\}=\sum_{\nu=2}^{2m}a_{2m-\nu}\,y^\nu\,T_\nu(x)$$

where

$$(1.6) \qquad T_\nu(x)=(1+x)^\nu-\nu x-1\,.$$

1⁰ We begin by estimating

$$(1.7) \qquad \Re\{y^{2m}T_{2m}(x)\}=|y|^{2m}\sum_{\nu=2}^{2m}\binom{2m}{\nu}|x|^\nu\cos(2m\theta+\nu\varphi).$$

When, according to (1.4), $\varepsilon-\pi/2\leqslant 2m(\theta+\varphi)\leqslant\pi/2+\varepsilon$, $\varepsilon-\pi/2\leqslant 2m\theta+{}$ ${}+2\varphi\leqslant\pi/2-\varepsilon$ one can infer that there exists a positive constant $k_1=k_1(\varepsilon)$ so that $\cos(2m\theta+\nu\varphi)>k_1$ and so

$$(1.8) \quad |y|^{2m}\sum_{\nu=2}^{2m}\binom{2m}{\nu}|x|^\nu\cos(2m\theta+2\varphi)\geqslant k_2(\varepsilon)\,|y|^{2m}\sum_{\nu=2}^{2m}\binom{2m}{\nu}|x|^\nu \text{ with } k_2>0.$$

From (1.7) and (1.8) we get at once

(1.9) $$\Re\{y^{2m}T_{2m}(x)\} > C_3|y|^{2m}\operatorname{Max}\{|x|^2, |x|^{2m}\}$$

for all $|x|>0$, $|y|>y_0$ and $\theta$, $\varphi$ satisfying (1.4).

2° We still have to show that

(1.10) $$\Re\{z_1\} > C_4\Re\{y^{2m}T_{2m}(x)\}$$

for all $|x|>0$, $|y|>y_0$ and $\theta$, $\varphi$ satisfying (1.4).

If $\nu=2,3,\ldots,2m$ there follows from (1.6) $T_\nu(x)\sim\binom{\nu}{2}x^2$, $x\to 0$ and $T_\nu(x)\sim x^\nu$, $x\to\infty$; hence

(1.11) $$|T_\nu(x)| < C_5|T_{2m}(x)| \quad \text{for all } |x|>0$$

According to (1.3) this gives

$$\Re\{z_1\} > C_5\Re\{y^{2m}T_{2m}(x)\}\left(1-C_6\sum_{\nu=2}^{2m-1}\frac{|a_{2m-\nu}|}{|y|^{2m-\nu}}\right)$$

$$> C_7\Re\{y^{2m}T_{2m}(x)\}$$

for all $|x|>0$, $|y|>y_0$ and $\theta$, $\varphi$ satisfying (1.4), which proves the lemma in the case of the function $z_1$. That it also holds for $z_2$ is obvious from the definition of $z_2$.

*Remark.* Solving the inequalities (1.4) we get that $\theta$ belongs to $\varepsilon-\frac{\pi}{4m}\frac{m+1}{m-1}\leqslant\theta\leqslant\frac{\pi}{4m}\frac{m+1}{m-1}-\varepsilon$ and, changing $y$ in $-y$ in (1.7), we deduce that the domain $\varepsilon+\pi-\frac{\pi}{4m}\frac{m+1}{m-1}\leqslant\theta\leqslant\pi+\frac{\pi}{4m}\frac{m+1}{m-1}-\varepsilon$ is also allowed because of $\cos(2m\pi+2m\theta+\nu\varphi)=\cos(2m+\nu\varphi)$.

Proof of the theorem 1. 1° Again let $x=|x|e^{\varphi i}$, $y=|y|e^{\theta i}$ and put $t=yu$ in the expression for $J(y)$; this gives

(1.12) $$J(y)=ye^{-P(y)+yP'(y)}\int_M e^{P(uy)+P'(y)yu+P(y)-yP'(y)}\,du$$

where $M$ is the line $\arg u=\varphi$. The formula (1.12) holds when $-\pi/2<$ $<2m(\theta+\varphi)<\pi/2$ for, the right side integral is then a regular function of $y$.

We, now, apply the saddle-point method to the integral (1.12). Consider the saddle point which is real and given by $u=1$, and put $u=1+x$. Thus we get

$$J(y)=e^{-P(y)+yP'(y)}I(y)$$

where

$$I(y)=y\int_N e^{-(z_1-z_2)-z_2}\,dx.$$

$N$ is obtained by displacing the contour $M$ parallel to itself for $-1$, and

$$z_1\equiv P(y+xy)-P'(y)\,y(1+x)-P(y)+P'(y)\,y=\sum_{\nu=2}^{2m}\frac{P^{(\nu)}(y)}{\nu!}\,y^{\nu}x^{\nu},$$

$$z_2=\frac{P''(y)\,y^2}{2}\,x^2+y^{2m}x^{2m}.$$

Using Cauchy's theorem we can move $N$ parallel to itself to the origin Applying Taylor's theorem to $e^{-(z_1-z_2)}$ we get

$$I(y)=\tilde{J}(y)+\tilde{R}_n(y)$$

where

$$\tilde{J}(y)=y\int_N e^{-z_2}\sum_{p=0}^{2n}\frac{(-1)^p}{p!}(z_1-z_2)^p\,dx,\quad \tilde{R}_n(y)=y\int_N R_{2n+1}(x;y)\,dx$$

(1.13)

$$R_{2n+1}(x;y)=\frac{1}{(2n)!}\int_0^1(1-t)^{2n}(z_1-z_2)^{2n+1}\,e^{-z_2-(z_1-z_2)\,t}\,dt.$$

It is easily verified that

$$\tilde{J}(y)=\sum_{k=0}^{(2m-1)n}A_{2k}(y)\,\Omega_{2k}(y) \tag{1.14}$$

and only the estimation of $\tilde{R}_n(y)$ is still needed.

2° Using Cauchy's theorem we can turn the contour $N$ in the integral by means of which $R_n(y)$ is defined so that the assumptions (1.4) are fulfilled.

It is well known [2] that

$$|R_{2n+1}(x;y)|\leqslant\frac{|z_1-z_2|^{2n+1}}{(2n+1)!}\,e^{-\mathrm{Min}\,(\Re\{z_1\},\,\Re\{z_2\})}\,. \tag{1.15}$$

But, according to (1.3) we have

$$|z_1-z_2|^{2n+1}\leqslant C_8\,|y|^{2m(2n+1)}\,\mathrm{Max}\,\{|x|^{3(2n+1)},\ |x|^{(2m+1)(2n+1)}\},\ |y|>y_0 \tag{1.16}$$

so that from (1.15), (1.16) and lemma 1 we deduce that

$$|R_{2n+1}(x;y)|\leqslant\frac{C}{(2n+1)!}\,|y|^{2m(2n+1)}\,\mathrm{Max}\,\{|x|^{3(2n+1)},\ |x|^{(2m+1)(2n+1)}\}$$

(1.17)

$$\cdot\exp\left[-C_0\,|y|^{2m}\,\mathrm{Max}\,\{|x|^2,\ |x|^{2m}\}\right],\ |y|>y_0.$$

It is, now, easy to estimate $\tilde{R}_n(y)$ by the use of (1.17). We have:

$$|\tilde{R}_n(y)| < 2|y| \int_0^\infty |R_{2n+1}(x; y| dx = \int_0^1 + \int_1^\infty .$$

From (1.17) follows

$$\int_0^1 < \frac{C_9 |y|^{m(2n+2)+1}}{(2n+1)!} \int_0^{C_0|y|^{2m}} u^{3n+1} e^{-u} du. \tag{1.18}$$

Hence, using Stirling's formula, we deduce, for all $n \geqslant 1$, on the one hand

$$\int_0^1 < C_{10} n^{\alpha_1 n} |y|^{-m(2n+2)+1} \tag{1.19}$$

and on the other

$$\int_0^1 < C_{11} n^{-\beta_1 n} |y|^{2m(2n+1)+1}. \tag{1.20}$$

Again, from (1.17) arguing as above we get for all $n \geqslant 1$

$$\int_1^\infty < C_{12} n^{\alpha_2 n} |y|^{-m(2n+2)+1} \tag{1.21}$$

and

$$\int_1^\infty < C_{13} |y|^{2n+1} n^{-\beta_2 n} \tag{1.22}$$

where $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ are some positive constants.

Formulae (1.19)—(1.22) give together

$$|\tilde{R}_n(y)| \leqslant \begin{cases} C_1 |y|^{-m(2n+2)+1} n^{\alpha n} \\ C_2 |y|^{2m(2n+1)+1} n^{-\beta n} \end{cases} \qquad n \geqslant 1.$$

According to the remark made at the end of the proof of lemma 1, the above inequalities hold when $y$ belongs to the domains $D_1$, $D_2$; this completes the proof.

**2.1.** In the sequel

$$f(x) \sim \sum \Phi_\nu(x)$$

stands for $\Phi_{\nu+1}(x) = 0(\Phi_\nu(x))$, $x \to \infty$ and $f(x) = \sum_{\nu=0}^{N-1} \Phi_\nu(x) + O(\Phi_N(x))$

THEOREM 2. *There exist numbers* $g_\nu = g_\nu(a_1, a_2 \cdots a_{2m-1})$, *so that the expansion*

$$J(y) \sim e^{-P(y)+yP'(y)} \sum \frac{g_\nu}{y^{m-1+\nu}} \tag{2.1}$$

*holds when*

$$-\frac{\pi}{4m}\,\frac{m+1}{m-1}+\varepsilon \leqslant \arg y \leqslant \frac{\pi}{4m}\,\frac{m+1}{m-1}-\varepsilon,$$

$$\varepsilon+\pi-\frac{\pi}{4m}\,\frac{m+1}{m-1} \leqslant \arg y \leqslant \pi+\frac{\pi}{4m}\,\frac{m+1}{m-1}-\varepsilon.$$

Proof of the theorem 2. We apply to the right-side integral in (1.2) the same argument as to the integral $I(y)$ in the proof of theorem 1 with

$$z_1 = \frac{P''(y)\,y^2}{2}\,x^2 + y^{2m}x^{2m}, \quad z_2 = \frac{P''(y)\,y^2}{2}\,x^2.$$

This gives

$$\Omega_{2k}(y) = \sum_{p=0}^{N-1} \beta_{k,p}\, y^{-2k} \{P''(y)\}^{-(mp+k+1/2)} + 2y \int_{L_\varphi} R_N(x;y)\,dx \tag{2.2}$$

where $\beta_{k,p} = \frac{(-1)^p}{p} 2^{mp+k+1/2}\,\Gamma(mp+k+1/2)$. We estimate the above integral using formula (1.15) and the obvious inequalities

$$|z_1 - z_2|^N \leqslant C_{14}|y|^{2mN}|x|^{2mN}, \quad \Re\{z_1\},\ \Re(z_2) > C_{15}|y|^{2m}|x|^2, \quad |y| > y_0$$

holding when the conditions (1.4) are fulfilled. Thus we get

$$\left|2y\int_{L_\varphi} R_N(x;y)\,dx\right| < C_{16}(N)\,y^{-\delta}, \quad \delta = m(2mN-2N+2k+1)-1. \tag{2.3}$$

Now, from the theorem 1, (2.2), (2.3) and the asymptotic expansion

$$\{P''(y)\}^{-(mp+k+1/2)} \sim y^{-(m-1)(2mp+k+1)} \sum \frac{d_{l,p,k}}{y^l}$$

where $d_{l,p,k}$ can be calculated quite elemetary, we deduce the expansion (2.1) where $g_0 = \sqrt{2\pi}\{(2m-1)\,2m\}^{-1/2}$ and $g_\nu$ are calculated as all the expansion in question were convergent series.

**2.2.** We deduce the asymptotic expansion of $F(z)$ from theorem 2 by means of the following

LEMMA 2. *Let the function $y=y(z)$ be defined by the equation $z=Q(y)$ with $Q(y)=P'(y)$; then, there exist numbers $\lambda_\nu=\lambda_\nu(a_1,\ldots,a_{2m})$, $\lambda_0=1$, so that the expansion*

$$y\sim\left(\frac{z}{2m}\right)^{\frac{1}{2m-1}}\sum\lambda_\nu z^{-\frac{\nu}{2m-1}}$$

*holds in the whole z-plane cut along the negative part of the real axis.*

Proof of the lemma 2. From the definition of $y(z)$ there follows

$$y\sim\left(\frac{z}{2m}\right)^{\frac{1}{2m-1}},\quad z\to\infty.$$

Further, put $y=z_0+z_0\omega_0$ in $z=Q(y)$, where $z_0=\left(\frac{z}{2m}\right)^{\frac{1}{2m-1}}$ and $\omega_0\equiv\omega_0(z)\to 0$ $(z\to\infty)$ and apply Taylor's theorem. This gives

$$z=\sum_{\nu=0}^{2m-1}\frac{Q^{(\nu)}(z_0)}{\nu!}(z_0\omega_0)^\nu. \tag{2.4}$$

Now, from

$$z_0^\nu Q^{(\nu)}(z_0)\sim 2m(2m-1)\cdots(2m-\nu)z_0^{2m-1},\ z\to\infty$$

and (2.4) we have

$$\omega_0\sim-\frac{a_1}{2m}z_0^{-1},\ z\to\infty\quad\text{and so}\quad y\sim z_0+\lambda_1,\ z\to\infty,\ \lambda_1=-\frac{a_1}{2m}.$$

If we proceed as above with $y=z_1+\lambda_1\omega_1$, where $z_1=z_0+\lambda_1$, we get

$$\omega_1\sim\frac{\lambda_2}{\lambda_1}z_0^{-1},\ \text{and so}\ y\sim z_0+\lambda_1+\frac{\lambda_2}{z_0}.\ z\to\infty$$

where

$$\lambda_2=\frac{1}{2}\left(\frac{a_1}{2m}\right)^2(2m-2)-\frac{a_2(2m-2)}{2m-1}$$

and the lemma follows by induction.

We can, now, prove the following

THEOREM 3. *There exist numbers $\alpha_\nu=\alpha_\nu(a_1,a_2,\ldots,a_{2m-1})$, $D_\nu=D_\nu(a_1,\ldots,a_{2m-1})$ so that the expansion*

$$F(z)\sim\exp\left\{\sum_{\nu=0}^{2m-1}D_\nu z^{\frac{2m-\nu}{2m-1}}\right\}\sum\alpha_\nu z^{-\frac{m-1+\nu}{2m-1}} \tag{2.5}$$

*holds in the whole z-plane cut along the negative part of the real axis, where*

$$\alpha_0 = \sqrt{2\pi}\{(2m-1)\,2m\}^{-1/2} \sum_{i=0}^{2m-2} a_i\,(2m-i-1)\,\lambda_i\,,\quad D_0 = (2m-1)(2m)^{-\frac{2m}{2m-1}}$$

$\lambda_\nu$ *being defined in the lemma* 2.

Proof of the theorem 3. We get the expansion (2.5) from (2.1) using lemma 2 and the elementary rules for calculation with asymptotic expansions.

In order to show that (2.5) holds in the whole $z$-plane, notice that, according to theorem 1, (2.1) holds when $y$ belongs to $D_1$ i. e. especially when

$$-\frac{\pi}{2}\,\frac{1}{2m-1}-\varepsilon \leqslant \arg y \leqslant \frac{\pi}{2}\,\frac{1}{2m-1}+\varepsilon\,.$$

Hence, according to lemma 2, (2.5) holds when $-\frac{\pi}{2} \leqslant \arg z \leqslant \frac{\pi}{2}$. But, (2.1) also holds when $y$ belongs to $D_2$, i. e. the expansion

$$J(e^{-\pi i}y) \sim e^{-P(e^{-\pi i}y)+e^{-\pi i}y\,P(e^{-\pi i}y)} \sum \frac{g_\nu}{(e^{-\pi i}\,y)^{m-1+\nu}} \tag{2.6}$$

holds especially when

$$-\frac{\pi}{2}\,\frac{1}{2m-1}-\varepsilon \leqslant \arg\,(e^{-\pi i}\,y) \leqslant \frac{\pi}{2}\,\frac{1}{2m-1}+\varepsilon\,.$$

From the lemma 2 follows that the expansion

$$e^{-\pi i}\,y \sim \left(\frac{e^{-\pi i}\,z}{2m}\right)^{\frac{1}{2m-1}} \sum \lambda_\nu\,(e^{-\pi i}\,z)^{-\frac{\nu}{2m-1}} \tag{2.7}$$

holds in the whole $z$-plane cut along the positive part of the real axis. From (2.6) and (2.7) we get as before

$$F(e^{-\pi i}\,z) \sim \exp\left\{\sum_{\nu=0}^{2m-1} D_\nu\,(z e^{-\pi i})^{\frac{2m-\nu}{2m-1}}\right\} \sum \alpha_\nu\,(e^{-\pi i}\,z)^{-\frac{m-1+\nu}{2m-1}}$$

holding for all $z$ satisfying the inequality

$$-\frac{\pi}{2} \leqslant \arg\,(e^{-\pi i}\,z) \leqslant \frac{\pi}{2}$$

so that the expansion (2.5) holds in the left half of the $z$-plane cut along the negative part of the real axis, with $-z$ instead of $z$.

**2.2.1.** From the theorem 1 one can easily deduce the expansion of the fundamental solution of the general onedimensional parabolic differential equation with constant coefficients. More precisely: let $b_\nu (\nu = 1, 2, \dots n-1)$ denote arbitrary complex numbers and

$$L\left(\frac{\partial}{\partial x}\right) = \sum_{\nu=0}^{n-1} b_\nu \left(\frac{\partial}{\partial x}\right)^{n-\nu}$$

and let the equation

$$\left\{L\left(\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial t}\right\} U(x; t) = 0$$

be parabolic, i. e. $n = 2m$ and $b_0 (-1)^{m-1} \xi^{2m}$ positive definite for real $\xi$. Its fundamental solution is given by [7]

$$K(x, t) = \int_{-\infty}^{\infty} e^{tL(iu) + ixy}\, du.$$

If in $F(z)$ for $a_\nu$ we choose especially $-b_\nu i^{2m} t^{\frac{\nu}{2m}}$ and $z = ixt^{-1/2m}$, then we get

$$K(x, t) = \frac{t^{-\frac{1}{2m}}}{2\pi} F(ixt^{-1/2m}), \quad \Re\{t\} > 0$$

From this formula, applying theorem 1, we deduce the wanted expansion; for the sake of completeness we frame the following

COROLLARY 1. *There exist numbers* $\alpha_\nu^* (x; b_1, \dots, b_{2m-1})$ *so that the expansion*

$$K(x, t) \sim t^{-\frac{1}{2(2m-1)}} \exp\left\{A_1 t^{-\frac{1}{2m-1}}\right\} \sum \alpha_\nu^* t^{\frac{\nu}{2m-1}}, \quad t \to 0$$

*holds when* $\Re\{t\} > 0$, *where,* $A_1 = x^{\frac{2m}{2m-1}} e^{\frac{i\pi}{2} \frac{2m}{2m-1}} (2m-1)\, 2m^{-\frac{2m}{2m-1}}$.

For the detailed proof of this corollary se [6].

**3.1.** In the sequel $M$ stands for an arbitrarily large, but fixed positive real number. We give in this section the representation of the $F\{P'(y)\}$ mentioned in § 1.1.

THEOREM 4. *Let the functions* $J(y)$, $\Omega_{2k}(y)$, $A_{2k}(y)$ *be defined as in* § 2.1, *then for all* $y \neq 0$ *we have*

$$J(y) = e^{-P(y) + yP'(y)} \sum_{k=0}^{\infty} A_{2k}(y)\, \Omega_{2k}(y).$$

Firstly we prove the following

LEMMA 3. *The series*

$$\sum_{k=0}^{\infty} A_{2k}(y)\,\Omega_{2k}(y)$$

*converges absolutely and uniformly when* $|y| \leqslant M$.

Proof of the lemma 3. 1° We begin by proving that there exists a constant $B > 1$ so that all the coefficients $b_\nu^{(i,p)}$ of the polynomials

$$Q_{i,p}(y) = \sum_{\nu=0}^{2mp-i} b_\nu^{(i,p)} y^{2mp-\nu}$$

satisfy

(3.1) $\qquad |b_\nu^{(i,p)}| < B^p \quad \text{when} \quad i = 3p+1,\ 3p+2,\ \ldots,\ (2m-1)p.$

Since $Q_{i,p}(y)$ are defined by (1.1) as the coefficients in the Taylor's expansion of the function

$$\{f(x,y)\}^p = \left\{\sum_{\nu=3}^{2m-1} \frac{P^{(\nu)}(y)\,y^\nu}{\nu!}\,x^\nu\right\}^p$$

they can be expressed by Cauchy's integral. Hence, since $|f(x,y)|^p < C_{17}|y|^{2mp}\cdot|x|^{(2m-1)p}$, $|x| \geqslant 1$, $|y| \geqslant 1$, we get $|Q_{i,p}(y)| \leqslant B^p|y|^{2mp}$. The assertion (3.1) then follows by repeating the above argument with $b_\nu^{(i,p)}$ instead of $Q_{i,p}(y)$ and $Q_{i,p}(y)$ instead of $\{f(x;y)\}^p$.

2° Let

$$S_n(y) = \sum_{k=0}^{n} |A_{2k}(y)|\,|\Omega_{2k}(y)|.$$

We show that $S_n(y) \to S(y)$, $n \to \infty$ uniformly.

Let

(3.2) $$\tilde{S}_{(2m-1)n}(y) = \sum_{k=2}^{(2m-1)n} \tilde{A}_{2k}(y)\,|\Omega_{2k}(y)|,\quad \tilde{A}_{2k}(y) = \sum_{p=1}^{\left[\frac{2k}{3}\right]} \frac{1}{p!}|\Omega_{2k}(y)|.$$

It is then easily verified that

(3.3) $$\tilde{S}_{(2m-1)n}(y) = \sum_{p=1}^{2n} \frac{a_p(y)}{p!} \quad \text{where} \quad a_p(y) = \sum_{i=3p}^{(2m-1)p} |Q_{i,p}(y)|\,|\Omega_i(y)|.$$

From the definition of $\Omega_i(y)$ we get if $|y| \leqslant M$

(3.4) $$|\Omega_i(y)| \leqslant C_{18}(M)\,2^{\frac{i}{2m}}\,\Gamma\left(\frac{i+1}{2m}\right)|y|^{-i}$$

where $k = k(M)$, $\alpha = \{2(m-1)\}^{-1}$, and from the definition of $Q_{i,p}(y)$ and (3.1) we get

(3.5) $$|Q_{i,p}(y)| \leqslant (2m-i+1)B^p \operatorname*{Max}_{0<|y|\leqslant M} \{|y|^i,\ |y|^{2mp}\}.$$

From (3.3), (3.4) and (3.5) there follows

$$\frac{a_p(y)}{p!} \leqslant C_{19}(M)\, p^2\, 2^{\frac{2m-1}{2m}p} \frac{B^p}{\Gamma(p+1)}\, \Gamma\left(\frac{2m-1}{2m}p + \frac{1}{2m}\right) \operatorname*{Max}_{0<|y|\leqslant M} \{1, |y|^{(2m-3)p}\}.$$

Hence, using Stirling's formula we deduce that there exists a $\delta > 0$ such that

$$a_p(y)/p! < C_{20}(M)\, p^{-\delta p} \operatorname*{Max}_{0<|y|\leqslant M} \{1, |y|^{(2m-3)p}\}.$$

We have thus proved the uniform convergence of $\Sigma a_p/p!$ and according to (3.3) that of $\tilde{S}_{(2m-1)n}(y)$.

Since $|A_i(y)| < \tilde{A}_i(y)$ for all $i$ in question one can infer that the series $S_{(2m-1)n}(y)$ converge uniformly too.

But, the sequence $S_n(y)$ is steadily increasing and $S_n(y) \leqslant S_{(2m-1)n}(y)$ when $m \geqslant 2$, $n \geqslant 1$ and $|y| \leqslant M$, and so the sequence $S_n(y)$ converges uniformly. Hence the lemma follows.

P r o o f of the theorem 4. We have proved in lemma 3 that $\sum_{k=0}^{\infty} A_{2k}(y)\, \Omega_{2k}(y) = S(y)$, when $|y| \leqslant M$; convergence being absolute and uniform. On the other hand, according to the inequality $|R_n(y) < < C_2 |y|^{2m(2n+1)+1} n^{-\beta n}$ proved in theorem 1 for all $y$ belonging to the domains $D_1$, $D_2$, we have

$$J(y) = e^{-P(y) + y P'(y)}\, S(y) \tag{3.6}$$

in these domains. But, the above formula holds not only in the domains $D_1$, $D_2$ but also, by analytic continuation, in the whole domain of regularity of the right side function in (3.6). According to lemma 3 this proves the theorem.

*(Received 30.X.1958)*

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