AN EXTENSION OF THE CESARI-CAVALIERI INEQUALITY

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1. *INTRODUCTION.* In his book, Surface Area (Princeton, 1956) [1, Chapter VI, sec. 20], L. Cesari proved an inequality of basic importance in the theory of Fréchet surfaces and of variational problems connected with them. Since the Cavalieri principle for surfaces is a special case of this inequality, he called it the Cavalieri inequality. More recently, various inequalities of the same type have been developed and extensively used. Hence we shall refer to the inequality of Cesari as the Cesari-Cavalieri inequality. In [1] the inequality was proved for surfaces defined over a planar disk. More recently, the inequality has been extended to surfaces defined over compact two dimensional manifolds [2]. In this note the inequality is extended in another direction which is useful in the study of the recently developed Cesari-Cavalieri area of a surface [3].

Let Q be a compact, triangulated 2-manifold (with or without boundary) and let $T: Q \to E_N$ be a continuous mapping of Q into N dimensional Euclidean space ($N \geqslant 2$). Then, as in [1], T, Q define a Fréchet surface S. Let $[S]$ be the set of points in E_N occupied by the surface S. Let $f: [T] \to$ reals be a continuous real valued function defined over $[S]$. For a real number t, let $C(t)$, $D^-(t)$, $D^+(t)$ be respectively the sets of points of Q for which $f(T(p)) = t$, $f(T(p)) < t$, $f(T(p)) > t$. $C(t)$ is the *contour* corresponding to t under f and the length of its image in $T(Q)$ can be computed as in [1, sec. 20] by considering prime ends on the boundary of $D^-(t)$. The function $l_f(t)$ which represents this length for each t in the range of f and is zero outside this range is called a contour length function. It can be shown that $l_f(t)$ is Lebesgue measurable for $-\infty < t < \infty$. Assume in addition that there exists a constant $K > 0$ such that for every pair of

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points $p_1, p_2 \in S$, $|f(p_1)-f(p_2)| \leqslant K|p_1-p_2|$ where the absolute value signs denote distances in the spaces involved. Then the Cesari-Cavalieri inequality asserts that $\int_{-\infty}^{+\infty} l_f(t)\,dt \leqslant KL(S)$ where $L(S)$ is the Lebesgue area of S. This inequality holds for all Lipschitzian functions of constant K defined over an open set containing $[S]$. However, in certain cases, the class of Lipschitzian functions of constant K is too restrictive. It would be convenient to have the inequality hold for functions which are not Lipschitzian in the large but which are in a certain sense only locally Lipschitzian. Since the Cesari proof depends only on local arguments, it appears that this objective could be realized and that the class of functions for which the inequality holds can be substantially enlarged. In this note we first define a class of locally Lipschitzian functions which includes the class of Lipschitzian functions of constant K and we then show that the Cesari-Cavalieri inequality holds also for this larger class.

2. *LOCALLY LIPSCHITZIAN FUNCTIONS AND THE EXTENDED INEQUALITY.*

DEFINITION 1. *Let A be a closed subset of a compact metric space B with distance function denoted by $|p_1-p_2|$. The family $J_K(A)$ of Lipschitzian functions of constant K is the set of all real valued functions defined on an open set $G \supset A$ such that if $p_1, p_2 \in G$, then $|f(p_1)-f(p_2)| \leqslant K|p_1-p_2|$.*

DEFINITION 2. *Let A be a closed subset of a compact metric space B. The family $J_K^L(A)$ of locally Lipschitzian functions of constant K is the set of all real valued functions $\{f\}$ defined on an open set $G \supset A$ satisfying the following conditions. For every $\varepsilon > 0$, there exists a real number $K_\varepsilon > 0$. and an open set $G_\varepsilon \supset A$ such that $\liminf_{\varepsilon \to 0} K_\varepsilon \leqslant K$ and $|f(p_1)-f(p_2)| \leqslant K_\varepsilon |p_1-p_2|$ for every $p_1, p_2 \in G_\varepsilon$ with $|p_1-p_2| < \varepsilon$.*

Several elementary facts about these classes are immediately evident. (1) $J_K(A) \subset J_K^L(A)$; (2) all functions in $J_K^L(A)$ are continuous on A; (3) if $K' \leqslant K$ then $J_K^L(A) \supset J_{K'}^{L}(A)$; (4) by the compactness of A, there exists a number $M \geqslant K$ such that $J_K^L A \subset J_M(A)$.

THEOREM. *Let T be a continuous mapping from a 2-manifold Q into E_N which defines a Fréchet surface S with Lebesgue area $L(S)$. Let $f \in J_K^L(S)$ and let $l_f(t)$ be the corresponding contour length function. Then*

$$\int_{-\infty}^{+\infty} l_f(t)\,dt \leqslant K\,L(S).$$

Proof: Evidently if $L(S)=\infty$ or if $\int_{-\infty}^{+\infty} l_f(t)\,dt=0$, the inequality is trivial. Hence we need not consider these cases.

The proof depends upon six lemmas. Of these six, the first three depend only upon the continuity of the function f and the proofs are exactly the same as in [1] for disks and as generalized in [2] for 2-manifolds. Lemmas 4 and 5 are exactly the same as in [1, sec. 20]. To avoid repetition we shall list these lemmas without proof and refer the reader to [1] and [2] for proofs. Lemma 6 will be stated and proved.

LEMMA 1. *Let* $T: Q \to E_N$ *be a continuous map of* Q *into* E_N. *Let* f_n, $n=0,1,2,\dots$, *be real valued and continuous on a fixed neighborhood* G *of* $[S]$ *such that* $f_n(p) \geqslant f_0(p)$ *for all* $p \in G$ *and for* $n=1,2,\dots$ *and let* $\lim_{n\to\infty} f_n(p)=f_0(p)$ *uniformly in* G. *For each* $n=0,1,2,\dots$, *let* $l_n(t)$ *be the contour length function corresponding to* f_n. *Then* $l_0(t) \leqslant \liminf_{n\to\infty} l_n(t)$ *for all* t.

LEMMA 2. *Let* T, Q *be as above and let* $f(p)$ *be continuous and real valued on a fixed neighborhood* G *of* S *with corresponding contour length function* $l_f(t)$. *Then* $l_f(t) \leqslant \liminf_{\tau\to t-} l_f(\tau)$ *for all* t, $-\infty < t < \infty$.

LEMMA 3. *Let* $T_n: Q \to E_n$, $n=0,1,2,\dots$, *be a sequence of continuous mappings from* Q *into* E_N *such that* $\lim_{n\to\infty} T_n(q)=T_0(q)$ *uniformly for* $q \in Q$. *Let* f *be a continuous real valued function defined on a fixed neighborhood* G *of* $T_0(Q)$. *Then for all* t $l_0(t) \leqslant \liminf_{\tau\to t-}\{\liminf_{n\to\infty} l_n(\tau)\}$ *where* $l_n(\tau)$, $n=0,1,2,\dots$, *are the contour length functions corresponding to* f *for* T_n.

LEMMA 4. *Every real valued function* $F(t)$, $-\infty < F(t) < \infty$ *such that* $F(t)=\liminf_{\tau\to t-} F(\tau)$ *for* $a \leqslant t \leqslant b$ *is measurable for* $a \leqslant t \leqslant b$.

LEMMA 5. *If* Δ *is a triangle in* E_2 *where* E_2 *has a coordinate system* (u,v), *if* $\varphi(u,v)=au+bv+c$ *is a linear non constant function on* Δ, *if* t_1, t_2 *are the minimum and maximum respectively of* $\varphi(u,v)$ *in* Δ, *and if* $\lambda(t)$, $t_1 \leqslant t \leqslant t_2$ *is the length of the segment of* Δ *on which* $\varphi(u,v)=t$, *then* $(a^2+b^2)\cdot \text{area}\,\Delta = \int_{-\infty}^{+\infty} \lambda(t)\,dt$.

For the proofs of lemmas 1—5, see [1, sec. 20] and [2].

LEMMA 6. *Let* $\varepsilon, \eta > 0$ *and let* M *be a compact subset of* E_N. *Let* f *be a real valued function satisfying the following conditions. There exists an open set* $G_\eta \supset M$ *and a real number* $K_\eta > 0$ *such that* f *is defined on* G_η *and* $|f(p_1)-f(p_2)| \leqslant K_\eta |p_1-p_2|$ *if* $p_1, p_2 \in G_\eta$ *and* $|p_1-p_2| < \eta$. *Then there exists a piecewise linear function* φ *defined on* G *with* $|\varphi(p)-f(p)| < \varepsilon$, $p \in G$ *and such that* $|\text{grad}\,\varphi| < K_\eta + \varepsilon$ *on some open set* G'_η *where* $M \subset G'_\eta \subset G_\eta$.

Proof: We construct the set $G'_\eta \subset G_\eta$ as follows. Let $G_\eta \subset C \subset E_N$ where C is a hyper cube. Let C be subdivided into equal smaller cubes $\{C_i'\}$, $i=1, 2, 3, \ldots, m$, by hyperplanes parallel to the faces of C in such a way that all of the sub cubes intersecting M plus all sub cubes which have a face of any dimension in common with these lie in G_η. This can be achieved by making the sub cubes sufficiently small. For every cube C_i', intersecting M, let C_i be the set consisting of C_i' plus all cubes adjacent to it. Let $G'_\eta = (\bigcup_i C_i)^\circ$. Then $M \subset G'_\eta \subset G_\eta$ and G'_η is open. Assume further that the original subdivision was made in such a way that each C_i has diameter less than η. Hence if $p_1, p_2 \in C_i$ for any i, $|f(p_1)-f(p_2)| \leqslant K_\eta |p_1-p_2|$. Let a coordinate system be set up in C using its edges as axes. As in [1, p. 326] we define the mean value integral of order n at each point of G' to be $f^{(n)}(p) = \int_{x_N}^{x_N+1/n} \ldots \int_{x_1}^{x_1+1/n} f(x_1, x_2, \ldots, x_N)\, dx \ldots dx_N$ where n is chosen sufficiently large to insure that no points outside G_η arise in the integrand. By theorems in [1], $f^{(n)}(p)$ is continuous, has continuous derivatives with respect to each variable, $\lim_{n\to\infty} f^{(n)}(p) = f(p)$ uniformly in G'_η, and if $p_1, p_2 \in C_i$ for any i, $|f^{(n)}(p_1) - f^{(n)}(p_2)| \leqslant K_\eta |p_1 - p_2|$. Let n be chosen so that $|f^{(n)}(p) - f(p)| < \varepsilon/2, p \in G'_\eta$. Let the G'_η be subdivided into smaller cubes by equally spaced hyperplanes parallel to the sides of C in such a way that on each of the smaller cubes the functions $f^{(n)}$, $f^{(n)}_{x_1}, f^{(n)}_{x_2}, \ldots, f^{(n)}_{x_N}$ have oscillations less than ε/N. It is then possible to divide each of these cubes into disjoint simplices in such a way that each simplex of the subdivision containes edges parallel to all the coordinate axes. Let $\{\sigma_j\}$, $j=1,2,3,\ldots,k$ be the collection of these simplices. Define the function φ to be linear in each simplex σ_j and to have the same values at the vertices as $f^{(n)}$. Evidently for any $p \in G'_\eta$, $|\varphi(p) - f(p)| \leqslant |\varphi(p) - f^{(n)}(p)| + |f^{(n)}(p) - f(p)| < \varepsilon/2 + \varepsilon/N \leqslant \varepsilon$. Hence φ satisfies the first conclusion of the lemma. To verify the second conclusion we note that in any σ_j the partial derivatives $\varphi_{x_1}, \varphi_{x_2}, \ldots, \varphi_{x_N}$ are constant. However, by the mean value theorem, at convenient points on the sides of σ_j parallel to the axes $\varphi_{x_1}(p_1) = f^{(n)}_{x_1}(p_1)$, $\varphi_{x_2}(p_2) = f^{(n)}_{x_1}(p_2), \ldots, \varphi_{x_N}(p_N) = f^{(n)}_{x_N}(p_N)$. Let $p, p' \in \sigma_j$. Then $|\varphi_{x_1}(p') - f^{(n)}_{x_1}(p)| < \varepsilon/N, \ldots, |\varphi_{x_N}(p') - f_{x_N}(p)| < \varepsilon/N$. Hence

$$|\operatorname{grad} \varphi(p')| - |\operatorname{grad} f^{(n)}(p)| =$$

$$= |(\varphi_{x_1}(p'))^2 + \cdots + (\varphi_{x_N}(p'))^2|^{1/2} - |(f^{(n)}_{x_1}(p))^2 + \cdots + (f^{(n)}_{x_N}(p))^2|^{1/2} \leqslant$$

$$\leqslant |\varphi_{x_1}(p') - f^{(n)}_{x_1}(p)| + \cdots + |\varphi_{x_N}(p') - f^{(n)}_{x_N}(p)| < \varepsilon$$

for any $p, p' \in \sigma_j$. Hence $|\operatorname{grad} \varphi(p)| \leqslant |\operatorname{grad} f^{(n)}(p)| + \varepsilon$. However, since $|\operatorname{grad} f^{(n)}(p)|$ is the maximum of all the directional derivatives of $f^{(n)}$ at p, if $\mu > 0$,

$$|(f^{(n)}_{x_1}(x_1, \ldots, x_N))^2 + \cdots + (f^{(n)}_{x_N}(x_1, \ldots, x_N))^2|^{1/2} \leqslant$$

$$\leqslant \left| \frac{f^{(n)}(x_1+h_1, x_2+h_2, \ldots, x_N+h_N) - f^{(n)}(x_1, x_2, \ldots, x_N)}{(h_1^2+h_2^2+\cdots+h_N^2)^{1/2}} \right| + \mu \leqslant K_\eta + \mu$$

for $h_1, h_2, \ldots, h_N$ properly chosen. Hence $|\operatorname{grad} f^{(n)}(p)| \leqslant K_\eta$ and $|\operatorname{grad} \varphi| \leqslant K_\eta + \varepsilon$ for all points of G'_η.

The proof of the theorem follows from lemmas 1—6 in exactly the same manner as in [1, p. 328]. To avoid repetition we shall state the steps involved and refer the reader to the proof of Cesari for details. Assume that $\eta > 0$ and K_η, G_η are given so that $|f(p_1) - f(p_2)| \leqslant K_\eta |p_1 - p_2|$ if $|p_1 - p_2| < \eta$, $p_1, p_2 \in G_\eta$.

(a) Let $T\colon Q \to E_N$ be piecewise linear from a triangulation of Q into E_N and assume that the triangulation of Q is sufficiently fine to insure that the image of each triangle δ_i, $i = 1, 2, 3, \ldots, n$ is of diameter less than η. Assume that f is also piecewise linear over $[S]$. Then for each triangle $\Delta_i = T(\delta_i) \subset S$, $K_\eta\, \alpha(\Delta) \geqslant \int_{-\infty}^{+\infty} l_f^{\Delta}(t)\, dt$ where $\alpha(\Delta)$ is the area of Δ and $l_f^{\Delta}(t)$ is the contour length function defined only over Δ. Thus as in [1], $K_\eta L(S) \geqslant \sum_{i=1}^{n} \int_{-\infty}^{+\infty} l_f^{\Delta_i}(t)\, dt = \int_{-\infty}^{+\infty} l_f(t)\, dt$, where $l_f(t)$ is the contour length function defined by f.

(b) If T is piecewise linear and $f \in J_K^L([S])$, then by using lemma 6, there exists a sequence of piecewise linear functions defined over open sets $G_n \supset S$ with domains of linearity in G_n of diameter each less than η such that for each n, $|\varphi_n(p) - f(p)| < 1/n$, $|\operatorname{grad} \varphi_n(p)| < K_\eta + 1/n$ for all $p \in G_n$. Then by use of lemmas 1 and the Fatou theorem as in [1, p. 329 (b)] it is proved that $K_\eta L(S) \geqslant \int_{-\infty}^{+\infty} l_f(t)\, dt$.

(c) If $T\colon Q \to E_N$ is any continuous mapping and if $f \in J_K^L([S])$ then by exactly the same proof as in [1, p. 329 (c)] where we need only restrict ourselves to approximating piecewise linear surfaces with triangles of diameter less than η, we prove by the use of lemma 3 and Fatou's theorem that

$$K_\eta L(S) \geqslant \int_{-\infty}^{+\infty} l_f(t)\, dt.$$

Assume that $f \in J_K^L([S])$. Then by definition of $J_K^L([S])$, for each $\eta > 0$ there exists $K_\eta > 0$ such that $\liminf_{\eta \to 0} K_\eta \leqslant K$ and $K_\eta L(S) \geqslant \int_{-\infty}^{+\infty} l_f(t)\, dt$. Hence $K L(S) \geqslant \int_{-\infty}^{+\infty} l_f(t)\, dt$.

3. *AN EXAMPLE.* In [3] the author defines the Cesari-Cavalieri area of a surface S to be $K(S) = \sup_{f \in J_1[(S)]} \int_{-\infty}^{+\infty} l_f(t)\, dt$. This definition of area has the usual properties which one associates with a reasonable definition of area and coincides with the Lebesgue area at least for simple surfaces. However, if S is a surface of large Lebesgue area but of small diameter, then any $f \in J_1([S])$ for which $\int_{-\infty}^{+\infty} l_f(t)\, dt$ gives an approximation of $L(S)$ must of necessity not have large oscillations. Hence, f must have many small variations and the contour it defines will probably be of a complicated nature. However, if we define $K^*(S) = \sup_{f \in J_1^L[(S)]} \int_{-\infty}^{+\infty} l_f(t)\, dt$, then the functions f in this class may have large oscillations in the large since the functions need be only locally Lipschitzian.

For example, let S be the surface of revolution generated by revolving a differentiable curve C: $y = g(x)$ about the x axis, $a \leqslant x \leqslant b$. Define f as follows. For each circular section of S perpendicular to the x axis define $f(p)$ to be the arc length s_a^x of C from $(a, g(a))$ to $(x, g(x))$. It is easily seen that $L(S) = K^*(S) = \int_{-\infty}^{+\infty} l_f(t)\, dt$. However, $f \operatorname{non} \in J_1([S])$ if g is not linear since if $p_a = (a, g(a)), p_b = (b, g(b))$, then $|f(p_b) - f(p_a)| \geqslant s_a^b \geqslant |p_a - p_b|$. It can easily be seen that $f \in J_1^L([S])$ since locally the arc length and chord length of, C approximate each other. Thus in this case $K^*(S) = L(S)$ is given by an extremely simple function f but in general $K(S)$ can be approximated only by integrals defined by relatively complicated functions in $J_1([S])$.

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