ON THE STATISTICAL MECHANICS OF CONTINUOUS MEDIA

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SUMMARY — The statistical mechanics of a continuous medium, which is much simpler than for a fluid, has been treated by Kampé de Fériet [1, 2]. For a vibrating string, due to the linear character of the equations, the problem can be completely solved; in [1] Kampé de Fériet treats the case of the infinite string; in [2] he solves the case of a string with fixed ends. In the present note it is shown that the technique of Kampé de Fériet applies also to a few other cases, such as a vibrating bar, beam, etc.

INTRODUCTION

In Reference [2] Kampé de Fériet points out that all the results so far obtained in the theory of turbulence are not at all in the same close relation with the theoretical equations of fluid mechanics as the classical statistical mechanics bears to the Hamilton-Jacobi equations for a dynamical system having a finite number of degrees of freedom. For such a system the main feautures of statistical mechanics are the following: (a) definition of the phase space Ω; (b) proof of a uniqueness theorem; (c) definition of a measure in Ω, invariant under the transformation; (d) proof of the ergodic theorem. All the work done in what is called statistical theory of turbulence deals only with some kind of time averages; a convenient phase space has never been introduced and there is the lack of a uniqueness theorem and of any method leading to the definition of an invariant measure. For this reason Kampé de Fériet attempts to sketch a statistical mechanics for a continuous medium, much simpler than a fluid. For a vibrating string, due to the linear character of the equations, the problem can be completely solved; in [1] Kampé de Fériet treats the case of the infinite string and in [2] the case of a string with fixed ends. Below we shall briefly outline the technique used by Kampé de Fériet.

The main characteristic features of the statistical mechanics of a conservative dynamical system having a finite number of degrees of freedom and satisfying the Hamilton-Jacobi equations are the following:

(a) definition of a phase-space Ω; every state of the system is characterized by a point $\omega \in \Omega$, whose $2k$ coordinates are the Lagrangian parameters q_i and the conjugate moments p_i;

(b) definition of a measure m, in Ω, invariant under the transformation $T_t\,\omega$;

(c) proof of the uniqueness theorem: starting at the initial time $t=0$ from a given initial state $\omega=\omega_0$, all the subsequent or prior states described as the transformations of ω, $T_t\,\omega$, where t varies from $-\infty$ to $+\infty$, are perfectly well determined and describe, in Ω, a curve, the trajectory $\Gamma(\omega)$;

(d) proof of the ergodic theorem: the time average,

$$\lim_{T\to+\infty}(2T)^{-1}\int_{-T}^{+T}F[T_t\,\omega]\,dt,$$

computed along a trajectory $\Gamma(\omega)$ exists for almost all ω and is equall to the statistical average, $\overline{F(\omega)}=\int_\Omega F[\omega]\,dm$ if the transformation $T_t\,\omega$ of Ω into itself is metrically transitive.

In constructing a statistical mechanics of continuous media one must consider a system having an infinite number of particles and consequently an infinite number of degrees of freedom; thus one must take as a phase-space, Ω, something more complicated: a function space.

In his approach to the statistical mechanics of a vibrating string, Kampé de Fériet takes as the phase-space Ω the following function space: Ω is the set of all real valued functions $f(s)$ of real variable s, $-\infty<s<+\infty$, satisfying the four conditions: (i) $f(s)$ is periodic, (ii) $f(s)$ is continuous, (iii) $f(s)$ has a continuous derivative, (iv) $\int_0^{2l}f(s)\,ds=0$. Every function $f(s)$ satisfying these assumptions is called a point ω of Ω. Every state of the vibrating string is well defined by the knowledge of the function $f(s)$. Kampé de Fériet notes that Ω is a separable Banach space. Let us introduce a one parameter Abelian group of one to one transformations of Ω into itself: $T_{t'+t''}\,\omega=T_{t''}(T_{t'}\,\omega)$ and let us establish a one to one correspondence between the points of the trajectory $\Gamma(\omega)$ and the points of a circle C of radius $l\pi^{-1}$ (l = length of the string). Let us fix the position of a point on C by the length γ of the arc ending at this point ($0 \leqq \gamma \leqq 2l$); then we obtain a one to one correspondence $\gamma \rightarrow T_\gamma \omega$.

Consider the linear functionals:

$$Y[\omega \mid x] = \int_{-x}^{+x} f(s)\,ds = y(x, t);$$

$$P[\omega \mid x] = f(x) + f(-x) = p(x, t);$$

$$V[\omega \mid x] = f(x) - f(-x) = v(x, t),$$

and usig the technique of Wiener construct a product space $A = B \times C'$ where C is the circle, considered above, and B is any measure space. Let us define on A a real valued function $g(\beta, \gamma)$ with the following properties:

(i) for every γ, g is a measurable function of β;

(ii) for every β, g is periodic in γ; $g(\beta, \gamma + 2l) = g(\beta, \gamma)$;

(iii) for every β, g is a continuous function of γ;

(iv) for every β, g has a continuous derivative g'_γ;

(v) for every β, $\int_0^{2l} g(\beta, \gamma)\,d\gamma = 0$.

We define a measure m, in Ω by the measure μ in A. We say that a functional $F[\omega]$, defined over the phase-space Ω, belongs to the class L, if its Lebesgue integral, computed with the measure m, exists. The statistical average is defined as:

$$\overline{F[\omega]} = \int_\Omega F[\omega]\,dm,$$

and the time average as

$$M_t F[\omega] = \lim_{T \rightarrow +\infty} (2T)^{-1} \int_{-T}^{+T} F[T_t \omega]\,dt.$$

Kampé de Fériet shows that, in general, there is:

$$\overline{F[\omega]} = \int_B M_t F[\omega]\,dv,$$

i. e., the two kinds of averages are not equal. But they are equal if both are equal to zero, i. e., $M_t F[\omega] = \overline{F[\omega]} = 0$. This condition is satisfied since

$$M_t F[\omega] = (2l)^{-1} \int_0^{2l} \left[\int_0^l g(\beta, s + \gamma)\,d\phi(s) \right] d\gamma;$$

inverting the order of integration and taking account of (ii) and (v), the proof follows immediately. One more item should be mentioned, namely, that the only case in which we have metric transitivity for the transformation of the phase-space Ω into itself corresponds to the case that $g(\beta, \gamma)$ does not depend on β: $g(\beta, \gamma) = g(\gamma)$. Thus for every functional $F[\omega] \in L$ we can put

$$F[\omega] = \Phi(\gamma) ;$$

$$\overline{F[\omega]} = (2l)^{-1} \int_0^{2l} \Phi(\gamma)\, d\gamma = M_t F[\omega].$$

In the present note the author shows that the technique of Kampé de Fériet in [2] can be applied to a few other cases of vibrating elements. For the sake of simplicity the author briefly discusses each section of Kampé de Fériet's paper emphasizing the points where some generalizations have to be introduced.

1. *FUNDAMENTAL EQUATIONS*

Below, we shall give the equations for some vibrating systems together with the corresponding boundary conditions.

(a) Vibrating string with fixed ends:

(1.1) $y''_{,tt}(x, t) = y''_{,xx}(x,t);$

(1.1*a*) $y(0, t) = y(l, t) = 0$ for every t;

(1.1*b*) $y'_{,t}(0, t) = y'_{,t}(l, t) = 0$ for every t.

(b) Longitudinal vibrations in a bar with both ends free:

(1.2) $y''_{,tt}(x, t) = y''_{,xx}(x, t);$

(1.2*a*) $y'_{,x}(0, t) = y'_{,x}(l, t) = 0$ for every t.

Organ pipe has the same equation.

(c) Longitudinal vibrations in a bar with one end fixed:

(1.3) $y''_{,tt}(x, t) = y''_{,xx}(x, t);$

(1.3*a*) $y(0, t) = y'_{,t}(0, t) = y'_{,x}(l, t) = 0$ for every t.

(d) Torsional vibrations of a uniform shaft with one end fixed:

(1.4) $$y''_{,tt}(x,t) = y''_{,xx}(x,t);$$

(1.4a) $$y(0,t) = y'_{,t}(0,t) = y'_{,x}(l,t) = 0 \quad \text{for every } t.$$

(e) Vibrating beam of uniform cross section with simply supported ends:

(1.5) $$y''_{,tt}(x,t) = -y^{(IV)}_{,xxxx}(x,t);$$

(1.5a) $$y(0,t) = y''_{,xx}(0,t) = y(l,t) = y''_{,xx}(l,t) = 0 \quad \text{for every } t.$$

(f) Vibrating cantilever (clamped free) beam of uniform cross section: like (e):

(1.6a) $$y(0,t) = y'_{,x}(0,t) = y''_{,xx}(l,t) = y'''_{,xxx}(l,t) = 0 \quad \text{for every } t.$$

(g) Vibrating beam with both ends clamped: like (e):

(1.7a) $$y(0,t) = y''_{,xx}(0,t) = y'''_{,xxx}(0,t) = y(l,t) = y''_{,xx}(l,t) = y'''_{,xxx}(l,t) = 0 \quad \text{for every } t.$$

2. *SOLUTIONS*

In Section 2 Kampé de Fériet derives the solution of the equation for the vibrating string. This will be given briefly below. Next, the author will derive the solutions of the equations (b) to (g), given in Section 1.

(a) *Vibrating string.*

Consider a vibrating string as a material curve moving in a plane Oxy; at rest, the string is a straight line segment, $0 \leqq x \leqq l$. Each point of the string moves parallel to the Oy axis; let $y(x,t)$ represent its displacement at the time t. The following four assumptions are made:

(2.1) $y(x,t)$ is a continuous function of (x,t):

(2.2) $y(0,t) = y(l,t) = 0$ for every t;

(2.3) $y(x,t)$ has continuous derivatives up to the second order;

(2.4) $y(x,t)$ is a solution of $y''_{,xx}(x,t) = y''_{,tt}(x,t)$.

We use the notations:

(2.5) $$p(x,t) = y'_{,x}(x,t); \quad v(x,t) = y'_{,t}(x,t); \quad j(x,t) = y''_{,tt}(x,t);$$

v and j represent respectively the velocity and acceleration of the point x at the time t. One may show that the motion of the string, from

$t=-\infty$ to $t=+\infty$, is completely determined by the knowledge of the displacement and the velocity of each point x at any time t. Namely, introduce two functions of x, $y(x)$ and $v(x)$, such that:

(2.6) the displacement $y(x)$ is a continuous function for $0 \leqq x \leqq l$;

(2.7) $$y(0)=y(l)=0;$$

(2.8) the first derivatives $y'(x)$ and $y''(x)$ exist and are continuous;

(2.9) the velocity $v(x)$ is a continuous function for $0 \leqq x \leqq l$;

(2.10) $$v(0)=v(l)=0;$$

(2.11) the first derivative $v'(x)$ exists and is continuous.

Construct a function $f(x)$ defined for $-\infty < x < +\infty$ by

(2.12) $$f(x)=\frac{1}{2}[y'(x)+v(x)], \qquad 0 \leqq x \leqq l,$$

(2.13) $$f(x)=\frac{1}{2}[y'(-x)-v(-x)], \qquad -l \leqq x \leqq 0,$$

(2.14) $$f(x+2l)=f(x).$$

The function $f(x)$ is a continuous function of x, having a continuous derivative $f'(x)$; also:

(2.15) $$\int_0^{2l} f(x)\,dx=0.$$

The unique displacement function $y(x,t)$, satisfying (2.1) to (2.4) and such that $y(x,\tau)=y(x)$; $v(x,\tau)=v(x)$, is given by the formula:

(2.16) $$y(x,t)=\int_{-x}^{+x} f(t-\tau+s)\,ds,$$

from which one may derive:

(2.17) $$p(x,t)=f(t-\tau+x)+f(t-\tau-x);$$

(2.18) $$v(x,t)=f(t-\tau+x)-f(t-\tau-x);$$

(2.19) $$j(x,t)=f'(t-\tau+x)-f'(t-\tau-x).$$

(b) *Longitudinal vibrations in a bar with both ends free.*

Assume the origin of the coordinate system in the center of the length of the bar. The following five assumptions are made:

(2.20) $y(x, t)$ is a continuous function of (x, t);

(2.21) $y(0, t) = 0$ for every t;

(2.22) $y(x, t)$ has continuous derivatives up to the second order;

(2.23) $y'_{,x}\left(-\frac{1}{2}l, t\right) = y'_{,x}\left(+\frac{1}{2}l, t\right) = 0$ for every t;

(2.24) $y(x, t)$ is a solution of $y''_{,tt}(x, t) = y''_{,xx}(x, t)$.

Introduce two functions of x, $y(x)$ and $v(x)$ such that:

(2.25) the displacement $y(x)$ is a continuous function of x for $-\frac{1}{2}l \leqq x \leqq \frac{1}{2}l$;

(2.26) $$y(0) = 0;$$

(2.27) the first derivatives $y'(x)$ and $y''(x)$ exist and are continuous and

$$y'\left(-\frac{1}{2}l\right) = y'\left(\frac{1}{2}l\right) = 0;$$

(2.28) the velocity $v(x)$ is a continuous function for $-\frac{1}{2}l \leqq x \leqq \frac{1}{2}l$;

(2.29) $$v(0) = 0;$$

(2.30) the first derivative $v'(x)$ exists and is continuous.

From these functions we derive a function $f(x)$ defined for $-\infty < x < +\infty$ by the conditions:

(2.31) $$f(x) = \frac{1}{2}[y'(x) + v(x)], \quad 0 \leqq x \leqq \frac{1}{2}l;$$

(2.32) $$f(x) = \frac{1}{2}[y'(-x) - v(-x)], \quad -\frac{1}{2}l \leqq x \leqq 0;$$

(2.33) $$f(x + 2l) = f(x).$$

The function $f(x)$ is a continuous function of x, having a continuous derivative $f'(x)$; also:

$$\int_0^{2l} (x)\, dx = 0.$$

Then the function $y(x,t)$, satisfying (2.20) to (2.24) and such that

$$y(x,\tau)=y(x), \quad v(x,\tau)=v(x),$$

is given by the formula:

$$y(x,t)=\int_{-x}^{+x} f(t-\tau+s)\,ds. \tag{2.34}$$

From (2.34) one can easily derive (2.17) to (2.19), which are also valid in the present case.

(c,d) *Longitudinal vibrations in a bar with one end fixed and torsional vibrations of a uniform shaft with one end fixed.*

The following five assumptions are made:

(2.35) $y(x,t)$ is a continuous function of (x,t);

(2.36) $y(0,t)=0$ for every t;

(2.37) $y(x,t)$ has continuous derivatives up to the second order;

(2.38) $y'_{,x}(l,t)=0$ for every t;

(2.39) $y(x,t)$ is a solution of $y''_{,xx}(x,t)=y''_{,tt}(x,t)$.

Introduce two functions of x, $y(x)$ and $v(x)$ such that:

(2.40) the displacement $y(x)$ is a continuous function of x for $0 \leqq x \leqq l$;

$$y(0)=0; \tag{2.41}$$

(2.42) the first derivatives $y'(x)$ and $y''(x)$ exist and are continuous;

$$y'(l)=0; \tag{2.43}$$

(2.44) the velocity $v(x)$ is a continuous function for $0 \leqq x \leqq l$;

$$v(0)=0; \tag{2.45}$$

(2.46) the first derivative $v'(x)$ exists and is continuous.

As above we derive for $-\infty < x < +\infty$:

$$f(x)=\frac{1}{2}[y'(x)+v(x)] \quad \text{for} \quad 0 \leqq x \leqq l; \tag{2.47}$$

(2.48) $$f(x)=\frac{1}{2}[y'(-x)-v(-x)] \quad \text{for} \quad -l \leqq x \leqq 0;$$

(2.49) $$\lim_{\pm\varepsilon\to 0} f(x+2l\pm\varepsilon)=\lim_{\pm\varepsilon\to 0} f(x\pm\varepsilon).$$

The function $f(x)$ is a continuous function of x, having a continuous derivative $f'(x)$ in the interval $[-l \leqq x \leqq l]$; at $x=\pm l$ the function has a well and uniquely defined jump (step) from $+A$ to $-A$; also: $\int_0^{2l} f(x)\,dx=0$. The derivative $f'(x)$ is continuous at $x=\pm l$. It is always a non-negative function of x.

For $x=\pm l$ the magnitude of the function $f(x)$ is defined by

$$f(\pm l)=\lim_{\pm\varepsilon\to 0} f(\pm l\pm\varepsilon).$$

As above:

(2.50) $$y(x,t)=\int_{-x}^{+x} f(t-\tau+s)\,ds,$$

from which one obtains again (2.17) to (2.19).

(e) *Vibrating beam with simply supported ends.*

The following five assumptions are made:

(2.51) $y(x,t)$ is a continuous function of (x,t);

(2.52) $$y(0,t)=y(l,t)=0 \quad \text{for every } t;$$

(2.53) $y(x,t)$ has continuous derivatives up to the second order in t and up to the fourth order in x;

(2.54) $$y''_{,xx}(0,t)=y''_{,xx}(l,t)=0 \quad \text{for every } t;$$

(2.55) $y(x,t)$ is a solution of $y''_{,tt}(x,t)=-y^{(IV)}_{,xxxx}(x,t)$.

Introduce two functions of x, $y(x)$ and $v(x)$ such that:

(2.56) the displacement $y(x)$ is a continuous function of x for $0 \leqq x \leqq l$;

(2.57) $$y(0)=y(l)=0;$$

(2.58) the derivatives $y'(x)$, $y''(x)$, $y'''(x)$, $y^{(IV)}(x)$ exist and are continuous;

(2.59) $$y''(0)=y''(l)=0; \quad y^{(IV)}(x)=-y''(x);$$

(2.60) the velocity $v(x)$ is a continuous function of x for $0 \leqq x \leqq l$,

$$v(0) = v(l) = 0;$$

(2.61) the derivatives $v'(x)$, $v''(x)$, $v'''(x)$, exist and are continuous;

(2.62) $$v'(0) = v'(l) = 0;$$

(2.63) $$v'''(x) = -v'(x).$$

As above, for $-\infty < x < +\infty$:

(2.64) $$f(x) = \frac{1}{2}[y'(x) + v(x)] \quad \text{for} \quad 0 \leqq x \leqq l;$$

(2.65) $$f(x) = \frac{1}{2}[y'(-x) - v(-x)] \quad \text{for} \quad -l \leqq x \leqq 0;$$

(2.66) $$f(x + 2l) = f(x);$$

(2.67) $$f'''(x) = -f'(x) \quad \text{for} \quad -l \leqq x \leqq l.$$

As is seen, $f(x)$ is a continuous function of x, having continuous derivatives f', f'', f''', and such that:

$$\int_0^{2l} f(x)\,dx = 0.$$

The unique displacement function $y(x, t)$ is given by the formula:

(2.68) $$y(x, t) = \int_{-x}^{+x} f(t - \tau + s)\,ds,$$

from which one obtains immediately:

(2.69) $y'_{,x}(x, t) = p(x, t) = f(t - \tau + x) + f(t - \tau - x);$

(2.70) $y''_{,xx}(x, t) = r(x, t) = f'(t - \tau + x) - f'(t - \tau - x);$

(2.71) $y'''_{,xxx}(x, t) = s(x, t) = f''(t - \tau + x) + f''(t - \tau - x);$

(2.72) $y^{(IV)}_{,xxxx}(x, t) = u(x, t) = f'''(t - \tau + x) - f'''(t - \tau + x) = -y''_{,xx}(x, t);$

(2.73) $y'_{,t}(x, t) = v(x, t) = f(t - \tau + x) - f(t - \tau - x);$

(2.74) $y''_{,tt}(x, t) = j(x, t) = f'(t - \tau + x) - f'(t - \tau - x).$

One can verify the boundary conditions:

(2.75) $y''_{,xx}(0, t) = \frac{1}{2}[y''(0)+v'(0)] - \frac{1}{2}[y''(0)-v'(0)] = v'(0) = 0$, etc.

(f) *Vibrating cantilever (clamped - free) beam.*

The following four assumptions are made:

(2.76) $y(x, t)$ is a continuous function of (x, t);

(2.77) $y(x, t)$ has continuous derivatives up to the second order in t and up to the fourth order in x;

(2.78) $y(0, t) = y'_{,x}(0, t) = y''_{,xx}(l, t) = y'''_{,xxx}(l, t) = 0$ for every t;

(2.79) $y(x, t)$ is a solution of $y''_{,tt}(x, t) = -y^{(IV)}_{,xxxx}(x, t)$.

Introduce two functions of x, $y(x)$ and $v(x)$ such that:

(2.80) the displacement $y(x)$ is a continuous function of x for $0 \leqq x \leqq l$

(2.81) $y(0) = 0$;

(2.82) the derivatives $y'(x)$, $y''(x)$, $y'''(x)$, $y^{(IV)}(x)$ exist and are continuous;

(2.83) $y'(0) = y''(l) = y'''(l) = 0$; $y^{(IV)}(x) = -y''(x)$;

(2.84) the velocity $v(x)$ is a continuous function of x for $0 \leqq x \leqq l$;

(2.85) $v(0) = 0$;

(2.86) the derivatives $v'(x)$, $v''(x)$, $v'''(x)$ exist and are continuous;

(2.87) $v'(l) = 0$; $v'''(x) = -v'(x)$.

From these functions one derives a function $f(x)$ defined for $-\infty < x < +\infty$ by:

(2.88) $f(x) = \frac{1}{2}[y'(x)+v(x)] \qquad 0 \leqq x \leqq l$;

(2.89) $f(x) = \frac{1}{2}[y'(-x)-v(-x)], \quad -l \leqq x \leqq 0$;

(2.90) $\lim_{\pm\varepsilon\to 0} f(x+2l\pm\varepsilon) = \lim_{\pm\varepsilon\to 0} f(x\pm\varepsilon)$;

(2.91) $f'''(x) = -f'(x) \quad$ for $\quad -l \leqq x \leqq l$.

Thus, $f(x)$ is a continuous function of x, having continuous derivatives f', f'', f''' in the interval $[-l \leqq x \leqq l]$; at $x = \pm l$ the function itself has a well and uniquely defined jump (step) from $+A$ to $-A$; also:

$$\int_0^{2l} f(x)\, dx = 0.$$

The derivatives f', f'', f''' are continuous at $x = \pm l$ and everywhere in $-\infty < x < +\infty$. The derivative f' is always a non-negative function of x. For $x = \pm l$, the magnitude of the function $f(x)$ is defined by

$$f(\pm l) = \lim_{\pm \varepsilon \to 0} f(\pm l \pm \varepsilon).$$

The unique displacement function $y(x, t)$, satisfying (2.76) to (2.79) and such that

$$y(x, \tau) = y(x); \quad v(x, \tau) = v(x),$$

is given by the formula:

(2.92)
$$y(x, t) = \int_{-x}^{+x} f(t - \tau + s)\, ds,$$

from which one obtains immediately equations (2.69) to (2.74). The boundary conditions are satisfied:

$$y'(0) = \frac{1}{2}[y'(0) + v(0)] + \frac{1}{2}[y'(0) - v(0)] = y'(0) = 0;$$

$$y''(l) = \frac{1}{2}[y''(l) + v'(l)] - \frac{1}{2}[y''(l) - v'(l)] = v'(l) = 0;$$

$$y'''(l) = \frac{1}{2}[y'''(l) + v''(l)] + \frac{1}{2}[y'''(l) - v''(l)] = y'''(l) = 0;$$

$$v'(l) = \frac{1}{2}[y''(l) + v'(l)] + \frac{1}{2}[y''(l) - v'(l)] = y''(l) = 0.$$

(g) *Vibrating beam with both ends clamped.*

The following five assumptions are made:

(2.93) $y(x, t)$ is a continuous function of (x, t);

(2.94) $y(0, t) = y(l, t) = 0$ for every t.

(2.95) $y(x, t)$ has continuous derivatives up to the second order in t and up to the fourth order in x;

(2.96) $y''_{,xx}(0,t) = y'''_{,xxx}(0,t) = y''_{,xx}(l,t) = y'''_{,xxx}(l,t) = 0$ for every t;

(2.97) $y(x,t)$ is a solution of $y''_{,tt}(x,t) = -y^{(IV)}_{,xxxx}(x,t)$.

Introduce two functions of x, $y(x)$ and $v(x)$ such that:

(2.98) the displacement $y(x)$ is a continuous function of x for $0 \leqq x \leqq l$;

(2.99) $$y(0) = y(l) = 0;$$

(2.100) the derivatives $y'(x), y''(x), y'''(x), y^{(IV)}(x)$ exist and are continuous;

(2.101) $y''(0) = y'''(0) = y''(l) = y'''(l) = 0;\ y^{(IV)}(x) = -y''(x)$;

(2.102) the velocity $v(x)$ is a continuous function of x for $0 \leqq x \leqq l$;

(2.103) $$v(0) = v(l) = 0;$$

(2.104) the derivatives $v'(x), v''(x), v'''(x)$ exist and are continuous;

(2.105) $v'(0) = v''(0) = v'(l) = v''(l) = 0;\ v''(x) = -v'(x)$.

As above:

(2.106) $$f(x) = \frac{1}{2}[y'(x) + v(x)], \quad 0 \leqq x \leqq l;$$

(2.107) $$f(x) = \frac{1}{2}[y'(-x) - v(-x)], \quad -l \leqq x \leqq 0;$$

(2.108) $$f(x + 2l) = f(x);$$

(2.109) $$f'''(x) = -f'(x) \quad \text{for} \quad -l \leqq x \leqq l.$$

The rest of the procedure remains the same.

3. *PHASE SPACE*

Beginning from this section we shall emphasize the generalizations which must be introduced in order to preserve the validity of Kampé de Fériet's proof in the cases considered. We take as phase space Ω the following function space: Ω is the set of all real valued functions $f(s)$ of a real variable s, $-\infty < s < +\infty$, satisfying the following five conditions:

(3.1) $f(s)$ is periodic, $f(s + 2l) = f(s)$ or

$$\lim_{\pm\varepsilon \to 0} f(s + 2l \pm \varepsilon) = \lim_{\pm\varepsilon \to 0} f(s \pm \varepsilon);$$

(3.2) $f(s)$ is continuous except possibly at a finite number of discrete points where the jump is well defined;

(3.3) $f(s)$ has continuous derivatives $f'(s), f''(s), f'''(s)$;

$$\int_0^{2l} f(s)\,ds = 0; \tag{3.4}$$

$$f'''(s) = -f'(s). \tag{3.5}$$

Every function $f(s)$ satisfying these assumptions is called a point ω of the space Ω. One may note that Ω is a separable Banach space.

4. *ABELIAN GROUP OF TRANSFORMATIONS*

In this section there is introduced a one parameter Abelian group of one to one transformations of Ω into itself:

$$\omega \to T_t\,\omega\,, \quad -\infty < t < +\infty\,; \tag{4.1}$$

ω corresponding to $f(s)$, one defines its transform $T_t\,\omega$ as the point corresponding to $f(t+s)$. Consider a fixed point $\omega \in \Omega$; the set of its transforms $T_t\,\omega$, when t varies from $-\infty$ to $+\infty$ is a closed curve which is called the trajectory Γ_ω. When t varies from $-\infty$ to $+\infty$, the transforms $T_t\,\omega$ of ω describe the trajectory Γ_ω infinitely many times. Or one can say: when t varies from $-\infty$ to $+\infty$, all the states of the vibrating string, corresponding to given initial conditions, are represented in the phase space Ω by the unique trajectory Γ_ω, which passes through the particular point ω representing the initial state. One can establish a one to one correspondence between the points of the trajectory Γ_ω and the points of a circle C of radius $l\pi^{-1}$; if one fixes the position of a point on C by the length of the arc ending at this point, $0 \leq \gamma < 2l$, then one gets a one to one correspondence:

$$\gamma \to T_t\,\omega\,. \tag{4.2}$$

5. *LINEAR FUNCTIONALS*

The symbol $F[\omega \,|\, \lambda_1, \ldots, \lambda_n]$ represents a functional defined on Ω, depending on n parameters; to every point $\omega \in \Omega$ and to every set of values of the parameters $\lambda_1, \ldots, \lambda_n$, corresponds a real number.

Consider the linear functionals:

$$Y[\omega \,|\, x] = \int_{-x}^{+x} f(s)\,ds\,, \quad 0 \leq x \leq l\,, \quad \text{or} \quad -\frac{1}{2}\,l \leq x \leq \frac{1}{2}\,l\,, \tag{5.1}$$

(5.2) $P[\omega|x]=f(x)+f(-x)$;

(5.3) $V[\omega|x]=f(x)-f(-x)$;

(5.4) $R[\omega|x]=f'(x)-f'(-x)$;

(5.5) $S[\omega|x]=f''(x)+f''(-x)$;

(5.6) $U[\omega|x]=f'''(x)-f'''(-x)=-R[\omega|x]$.

Along the trajectory Γ_ω one has:

(5.7) $Y[T_t\omega|x]=y(x,t)$; $P[T_t\omega|x]=p(x,t)$;

(5.8) $V[T_t\omega|x]=v(x,t)$; $R[T_t\omega|x]=r(x,t)$;

(5.9) $S[T_t\omega|x]=s(x,t)$; $U[T_t\omega|x]=u(x,t)=-r(x,t)$.

6. *INVARIANT MEASURE*

In this section Kampé de Fériet derives the invariant measure „m" in the phase space Ω. Below, there are given the generalizations which should be introduced to adjust Kampé de Fériet's proof to the present cases.

Let us take as an abstract space A a product space $A=B\times C$, where C is the circle considered in Section 4, and B is any measure space. Let us define on A a real valued function $g(\beta,\gamma)$, $\beta\in B$, $\gamma\in C$, with the following properties:

(6.1) for every γ, g is a measurable function of β;

(6.2) for every β, g is periodic in γ, $g(\beta,\gamma+2l)=g(\beta,\gamma)$;

(6.3) for every β, g is a continuous function of γ;

(6.4) for every β, g has a continuous derivative $g'_{,\gamma}(\beta,\gamma)$;

(6.5) for every β, $\int_0^{2l} g(\beta,\gamma)\,d\gamma=0.$

To this conditions we have to add:

(6.6) for every β, g has continuous derivatives $g''_{,\gamma\gamma}(\beta,\gamma)$, $g'''_{,\gamma\gamma\gamma}(\beta,\gamma)$ and $g'''_{,\gamma\gamma\gamma}=-g'_{,\gamma}$.

If we put

(6.7) $f(s)=g(\beta,s+\gamma)$, $\beta\in B,\gamma\in C$, the set of Ω^* of all these functions $f(s)$ is a subset of the phase space Ω, since the conditions (6.2) to (6.6) imply the conditions (3.1) to (3.5). The remaining part of the proof of Kampé de Fériet remains valid in all the cases discussed above, which shows that the invariant measure can be found in all the cases of vibrating elements discussed above. It is a simple matter to adjust all the remaining details of the proof to problems in question.

7. *STATISTICAL AND TIME AVERAGES*

In this section Kampé de Fériet shows that for general functionals the two kinds of averages are not equal. However, for any linear functional these averages are both equal to zero. This case refers to the vibrating string and to all the vibrating elements discussed in the present note.

In Section 8, Kampé de Fériet calculates moments and in Section 9 he shows that the only case in which we have metric transitivity for the transformation of the phase space Ω into itself corresponds to the case that $g(\beta, \gamma)$ does not depend on β: $g(\beta, \gamma) = g(\gamma)$.

FINAL REMARKS

Above we have shown that the sketch of a statistical mechanics for a continuous medium proposed by Kampé de Fériet and valid for a vibrating string can be readily generalized to a few other vibrating one dimensional elements.

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