ON THE INTEGRABILITY OF FUNCTIONS DEFINED BY COSINE SERIES WITH MONOTONE DECREASING COEFFICIENTS

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Let us consider the following trigonometric series with monotone decreasing coefficients $\lambda_n \downarrow 0$, $n \to \infty$,

$$f(x) = \frac{1}{2}\lambda_0 + \sum_{n=1}^{\infty} \lambda_n \cos nx, \quad g(x) = \sum_{n=1}^{\infty} \lambda_n \sin nx.$$

The functions defined by these series are continuous in $\delta \leqslant x \leqslant 2\pi - \delta$, for every $\delta > 0$, but are not necessarily L-integrable in $(0, \pi)$.

It is known, however, that, when $h(x) = f(x)$ or $g(x)$, we have the following theorem:

$$\eta(x)\, h(x) \in L\,(0, \pi) \textit{ if and only if } \sum_{n=0}^{\infty} \frac{\lambda_n}{n} \eta\left(\frac{1}{n}\right) < \infty,$$

where $\eta(x) = x^{-r}$ *or* $x^{-r} L(1/x)$ ($L(x)$ being a slowly increasing function in the sence of Karamata [5]) $0 < r < 1$ when $h(x) = f(x)$; $0 \leqslant r < 2$ when $h(x) = g(x)$. (For $\eta(x) = x^{-r}$, see Boas [2], Heywood [4], Sunouchi [8], and Young [9]; for $\eta(x) = x^{-r} L(1/x)$, see Aljančić, Bojanić and Tomić [1] and Peyerimhoff [7].)

Instead of $\eta(x) = x^{-r}$ or $x^{-r} L(1/x)$, a more general $\eta(x)$ is considered in Peyerimhoff [7], namely, $\eta(x) \geqslant 0$ and $x\,\eta(x) \in L\,(0, \pi)$, and the following result is obtained:

$$\eta(x)\, g(x) \in L\,(0, \pi) \text{ if } (\mathrm{I}) \ldots \sum_{n=1}^{\infty} n \lambda_n \int_0^{1/n} x\,\eta(x)\, dx < \infty.$$

The converse also holds if we have some additional condition, *e. g.*,

$$\text{(II)} \ldots\ldots \int_{1/n}^{\pi} \frac{\eta(x)}{x} \sin^2\left(n+\frac{1}{2}\right)\frac{1}{2}x\,dx \geqslant \varepsilon \int_{1/n}^{\pi} \frac{\eta(x)}{x}\,dx, \quad for \quad \begin{matrix} \varepsilon > 0 \\ n > N. \end{matrix}$$

The condition (II) is satisfied in the following cases: *For* $x_0 \leqslant \frac{\pi}{\lambda}$ $(\lambda > 1)$

$$\eta(y) \leqslant \alpha\eta(x) \quad for\ all \quad 0 < x \leqslant y \leqslant \lambda x, \quad x \leqslant x_0, \quad or$$

$$\eta(x) \leqslant \beta\eta(y) \quad for\ all \quad 0 < x \leqslant y \leqslant \lambda x, \quad x \leqslant x_0.$$

In many cases, the condition (I) can be replaced by

$$\text{(III)} \qquad \sum_{n=1}^{\infty} \frac{\lambda_n}{n} \eta\left(\frac{1}{n}\right) < \infty.$$

In the present work we shall establish parallel theorems concerning $f(x)$ as only the case of $g(x)$ is considered in Peyerimhoff [7]. We shall always assume the following conditions in this paper:

i) $\eta(x)$ *is a non-negative and L-integrable function in* $(0, \pi)$.

ii) $\{\lambda_n\}$ *is a monotone decreasing sequence tending to zero.*

iii) $f(x) = \frac{1}{2}\lambda_0 + \sum_{n=1}^{\infty} \lambda_n \cos nx.$

Summarizing, the following results are obtained:

THEOREM 1: *If*

$$\text{(1)} \qquad \sum_{n=1}^{\infty} \lambda_n \int_0^{1/n} \eta(x)\,dx < \infty$$

then $\eta(x) f(x) \in L(0, \pi)$.

THEOREM 2: *If* $\eta(x) f(x) \in L(0, \pi)$ *then*

$$\text{(2)} \qquad \int_t^{x_0} \frac{\eta(x)}{x}\,dx \leqslant C\eta(t) \qquad (0 < t \leqslant x_0 \leqslant \pi, \quad C: a\ constant)$$

implies (1).

THEOREM 3: *If* $\zeta(x)$ *satisfies the following condition:*

$$\text{(3)} \quad 0 < \zeta(y) \leqslant \lambda^s \zeta(x) \ for\ all\ 0 < x \leqslant y \leqslant \lambda x,\ x \leqslant x_0 \leqslant \pi/\lambda,\ s > 0,\ \lambda > 1.$$

then

$$\int_x^{x_0} \frac{\zeta(t)\,dt}{t^{1+s+\varepsilon}} \geqslant C\,\frac{\zeta(x)}{x^{s+\varepsilon}} \qquad (C: a\ constant,\ \varepsilon > 0) \tag{4}$$

holds.

THEOREM 4: *If* $\eta(x)$ *satisfies the following conditions:*

(5)
a) $\eta(y) \leqslant \alpha\,\eta(x)$ *for all* $0 < x \leqslant y \leqslant \lambda x,\ x \leqslant x_0 \leqslant \pi/\lambda,\ \alpha > 0,\ \lambda > 1,$

b) $\eta(x) \leqslant \beta\,\eta(y)$ *for all* $0 < x \leqslant y \leqslant \lambda x,\ x \leqslant x_0 \leqslant \pi/\lambda,\ \beta < \lambda,\ \lambda > 1,$

then the condition (1) *is equivalent to the condition* (III).

In Theorem 3, for $\zeta(x) = x^h \cdot L(1/x)\ (h = s+\varepsilon-r,\ s \geqslant r > \varepsilon)$, the condition (3) is satisfied since

$$\frac{\zeta(y)}{\zeta(x)} = \left(\frac{y}{x}\right)^h \frac{L(1/y)}{L(1/x)} \leqslant \lambda^h \cdot \frac{L(1/y)}{L(1/x)} < \lambda^s, \text{ as } x \to 0\ ^{*)} \qquad (0 < x \leqslant y \leqslant \lambda x).$$

And if $\eta(x) = x^{-s-\varepsilon}\zeta(x) = x^{-r}L(1/x)\ (0 < r < 1)$, then (2) holds because of (4). Moreover, for the above $\eta(x)$, the condition (5) is satisfied; it follows that we can derive the following theorem, from Theorems 1, 2, and 4:

$$x^{-r} L\left(\frac{1}{x}\right) f(x) \in L(0,\pi) \text{ holds, if and only if } \sum_{n=1}^{\infty} \frac{\lambda_n L(n)}{n^{1-r}} < \infty.$$

Therefore, our theorems include the previous result of Aljančić, Bojanić and Tomić [1].

To prove Theorem 1, we need the following:

LEMMA 1: *If* (1) *holds, then*

$$\sum_{n=1}^{\infty} n\,\Delta\lambda_n \int_0^{1/n} \eta(x)\,dx < \infty \tag{6}$$

and

$$\sum_{n=1}^{\infty} \Delta\lambda_n \int_{1/n}^{\pi} \frac{\eta(x)}{x}\,dx < \infty. \tag{7}$$

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*) Since $\dfrac{L(1/y)}{L(1/x)} \simeq 1$, as $x \to 0$, for $0 < x \leqslant y \leqslant \lambda x$.

Proof: Let us consider the following identity:

$$\sum_{n=1}^{N} n\, a_n \cdot \Delta\lambda_n + \sum_{n=1}^{N} \Delta\lambda_n \cdot \sum_{\nu=1}^{n-1} \nu\,(a_\nu - a_{\nu+1}) = \sum_{\nu=1}^{N} a_\nu\,(\lambda_\nu - \lambda_{N+1}). \tag{8}$$

((8) is obtained directly from

$$n\,a_n + \sum_{\nu=1}^{n-1} \nu\,(a_\nu - a_{\nu+1}) = \sum_{\nu=1}^{n} a_\nu,$$

and

$$\sum_{n=1}^{N} \sum_{\nu=1}^{n} a_\nu \Delta\lambda_n = \sum_{\nu=1}^{N} a_\nu\,(\lambda_\nu - \lambda_{N+1}).\Big)$$

For $a_\nu = \int_0^{1/n} \eta(x)\,dx$, the right hand side of (8) remains bounded for $N \to \infty$ because of (1). It follows that both terms on the left hand side are finite since all the terms are positive; hence (6) holds. We also obtain:

$$\sum_{n=1}^{\infty} \Delta\lambda_n \cdot \sum_{\nu=1}^{n-1} \nu \cdot \int_{1/(\nu+1)}^{1/\nu} \eta(x)\,dx < \infty.$$

Since

$$\frac{1}{2}\int_{1/(\nu+1)}^{1/\nu} \frac{\eta(x)}{x}\,dx \leqslant \nu \cdot \int_{1/(\nu+1)}^{1/\nu} \eta(x)\,dx \leqslant \int_{1/(\nu+1)}^{1/\nu} \frac{\eta(x)}{x}\,dx \quad (\nu \geqslant 1),$$

we obtain

$$\sum_{n=1}^{\infty} \Delta\lambda_n \int_{1/n}^{1} \frac{\eta(x)}{x}\,dx < \infty;$$

this implies (7) because $\eta(x)/x \in L(1, \pi)$.

We are, now, ready to prove Theorem 1.

Proof: Since

$$\int_0^{\pi} \eta(x)\,|f(x)|\,dx \leqslant \int_0^{\pi} \eta(x) \left\{\frac{1}{2}\Delta\lambda_0 + \sum_{n=1}^{\infty} \Delta\lambda_n \left|\frac{\sin(n+1/2)\,x}{2\sin x/2}\right|\right\} dx$$

$$\leqslant \frac{1}{2}\Delta\lambda_0 \int_0^{\pi} \eta(x)\,dx + \sum_{n=1}^{\infty} \Delta\lambda_n \int_0^{\pi} \eta(x) \left|\frac{\sin(n+1/2)\,x}{2\sin x/2}\right| dx,$$

it is sufficient to show

$$(9)\qquad \sum_{n=1}^{\infty} \Delta\lambda_n \int_0^{\pi} \eta(x) \left|\frac{\sin(n+1/2)x}{2\sin x/2}\right| dx < \infty,$$

and this follows from (6) and (7) since

$$\sum_{n=1}^{\infty} \Delta\lambda_n \left\{ \int_0^{1/n} + \int_{1/n}^{\pi} \right\} \eta(x) \left|\frac{\sin(n+1/2)x}{2\sin x/2}\right| dx$$

$$\leqslant \sum_{n=1}^{\infty} (2n+1)\Delta\lambda_n \int_0^{1/n} \eta(x)\,dx + 2\sum_{n=1}^{\infty} \Delta\lambda_n \int_{1/n}^{\pi} \frac{\eta(x)}{x} < \infty .$$

To prove Theorem 2, let us first consider the following

LEMMA 2: *If* $f(x) \in L(0,\pi)$ *and* $F(x) = \int_0^x f(x)\,dx$, *then*

$$(10)\qquad \frac{\eta(x)}{x} F(x) \in L(0,\pi)$$

implies (1).

Proof: Since $\eta(x) \in L(0,\pi)$, we have $\frac{\eta(x)}{x} F_1(x) \in L(0,\pi)$ for $F_1(x) = F(x) - {}^1/_2\lambda_0 x$. Writing $\mu_n = \lambda_n/n$ and applying Abel's transformation to $F_1(x)$ we obtain

$$F_1(x) = \sum_{n=1}^{\infty} \mu_n \sin nx = \sum_{n=1}^{\infty} \Delta\mu_n \frac{\sin^2(n+1/2)x/2}{\sin x/2} - \frac{1}{2}\mu_1 \tan\frac{1}{4}x.$$

We have $\frac{\eta(x)}{x} \tan x/4 \in L(0,\pi)$ since $x^{-1}\tan x/4$ is bounded in $(0,\pi)$, and therefore we have $(F_1(x) + {}^1/_2\,\mu_1 \tan x/4)\,\eta(x)/x \in L(0,\pi)$, i. e., (And since $\mu_n \downarrow 0,\ n \to \infty$,)

$$\int_0^\pi \sum_{n=1}^\infty \frac{\eta(x)}{x}\,\Delta\mu_n \frac{\sin^2 (n+1/2)\,x/2}{\sin x/2}\,dx = \sum_{n=1}^\infty \Delta\mu_n \int_0^\pi \frac{\eta(x)}{x}\cdot\frac{\sin^2 (n+1/2)\,x/2}{\sin x/2}\,dx < \infty.$$

Then, we obtain

$$\sum_{n=1}^\infty (n+1)^2\,\Delta\mu \int_0^{1/n} \eta(x)\,dx < \infty$$

since

$$(n+1)^2 \int_0^{1/n} \eta(x)\,dx \leqslant 4 \int_0^{1/n} \frac{\eta(x)}{x}\cdot\frac{\sin^2 (n+1/2)\,x/2}{\sin x/2}\,dx.$$

From $(n+1)^2\,\Delta\mu_n = (n+1)\,\Delta\lambda_n + (1+1/n)\,\lambda_n$, we obtain

$$\sum_{n=1}^\infty \left(1+\frac{1}{n}\right)\lambda_n \int_0^{1/n} \eta(x)\,dx < \infty;$$

hence (1) holds.

P r o o f: of Theorem 2 We assume $\eta(x) \neq 0$ almost everywhere in $(0, x_0)$; the proof is trivial otherwise. Then, we have from (2),

$$\eta(t) \geqslant \frac{1}{C}\int_t^{x_0} \frac{\eta(x)}{x}\,dx \geqslant \frac{1}{Cx_0}\int_t^{x_0} \eta(x)\,dx \geqslant \frac{1}{Cx_0}\int_\delta^{x_0} \eta(x)\,dx = \alpha > 0 \qquad (t \leqslant \delta)$$

i. e., there exists $\alpha > 0$ such that $\eta(t) \geqslant \alpha$ for $0 < t \leqslant \delta$.

It follows that $\alpha\,|f(x)| \leqslant \eta(x)\,|f(x)| \in L(0,\delta)$; therefore, $f(x) \in L(0,\delta)$. Since $f(x)$ is bounded in (δ,π), we have $f(x) \in L(0,\pi)$.

From $\eta(x) f(x) \in L(0,\pi)$, we obtain by (2)

$$\int_0^{x_0} |F(x)|\frac{\eta(x)}{x}\,dx \leqslant \int_0^{x_0} \frac{\eta(x)}{x}\,dx \int_0^x |f(t)|\,dt = \int_0^{x_0}\int_t^{x_0} \frac{\eta(x)}{x}\,dx\,|f(t)|\,dt$$

$$\leqslant C\int_0^{x_0} \eta(t)\,|f(t)|\,dt,$$

i. e., $\frac{\eta(x)}{x} F(x) \in L(0, x_0)$ and since $\frac{\eta(x)}{x}$ is bounded in (x_0, π), we obtain $\frac{\eta(x)}{x} F(x) \in L(0, \pi)$.

It follows that (1) holds from Lemma 2.

Proof of Theorem 3: Let $\lambda^n x \leqslant x_0 \leqslant \lambda^{n+1} x$, then we have for the constants C_i $(i=1,2,3,4)$ depending only on λ, s, and ε,

$$\int_x^{x_0} \frac{\zeta(t)\,dt}{t^{1+s+\varepsilon}} \leqslant \sum_{\nu=0}^{n} \int_{\lambda^\nu x}^{\lambda^{\nu+1} x} \frac{\zeta(t)\,dt}{t^{1+s+\varepsilon}}$$

$$\leqslant C_1 \sum_{\nu=0}^{n} \zeta(\lambda^\nu x) \cdot \{1/(\lambda^\nu x)^{s+\varepsilon} - 1/(\lambda^{\nu+1} x)^{s+\varepsilon}\}$$

$$\leqslant C_2 \sum_{\nu=0}^{n} \zeta(\lambda^\nu x)/(\lambda^\nu x)^{s+\varepsilon} \leqslant C_3 \left\{\sum_{\nu=0}^{n} 1/\lambda^{\varepsilon\nu}\right\} \cdot \zeta(x)/x^{s+\varepsilon}$$

$$\leqslant C_4 \zeta(x)/x^{s+\varepsilon}.$$

If (5) a) holds and if $1/(n\lambda^{\nu+1}) \leqslant x \leqslant 1/(n\lambda^\nu)$ $(1/n \leqslant x_0)$, then repeatedly applying (5) a), we get the following

$$\frac{1}{\alpha^{\nu+1}} \eta\left(\frac{1}{n}\right) \leqslant \eta(x).$$

And since

$$\int_0^{1/n} \eta(x)\,dx = \sum_{\nu=0}^{\infty} \int_{1/n\lambda^{\nu+1}}^{1/n\lambda^\nu} \eta(x)\,dx,$$

it follows that

$$A\,\eta\left(\frac{1}{n}\right)\frac{1}{n} \leqslant \int_0^{1/n} \eta(x)\,dx \quad \left(n > \frac{1}{x_0}\right) \tag{11}$$

where

$$A = (\lambda - 1) \cdot \sum_{\nu=1}^{\infty} (\alpha\lambda)^{-\nu}.$$

Similarly, from (5) b), we obtain

$$B\,\eta\left(\frac{1}{n}\right)\frac{1}{n}\geqslant\int_0^{1/n}\eta(x)\,dx\qquad\left(n>\frac{1}{x_0}\right)\tag{12}$$

where

$$B=(\lambda-1)\cdot\sum_{\nu=1}^{\infty}\left(\frac{\beta}{\lambda}\right)^{\nu}$$

Proof of Theorem 4: If (5) holds, then (11) and (12) hold for $n>1/x_0$, i.e.,

$$\int_0^{1/n}\eta(x)\,dx\sim\frac{1}{n}\,\eta\left(\frac{1}{n}\right),\ \text{as } n\to\infty.$$

Therefore the condition (1) and the condition (3) are equivalent since any finite sum of the series of (1) or of (III) is finite.

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